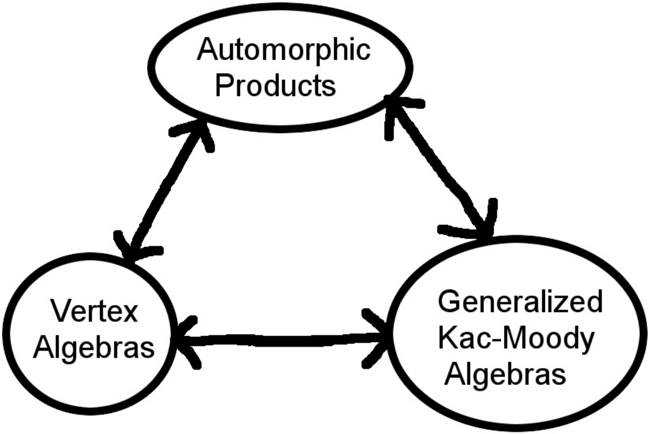
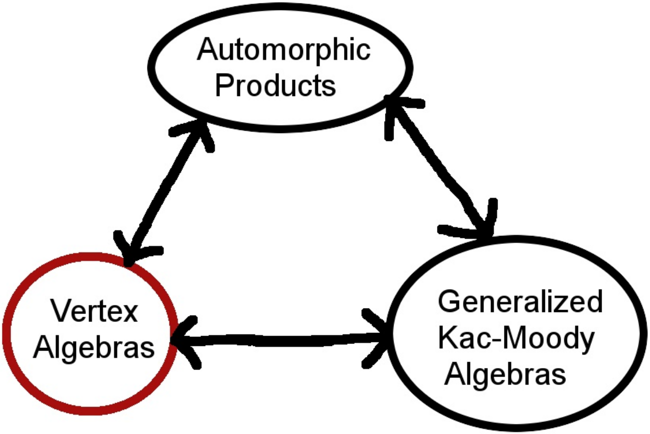
Vertex Algebra constructions of Generalized Kac-Moody Algebras — An overview

Gerald Höhn
Kansas State University

September 1, 2009


Automorphic
Products
Vertex
Algebras
Generalized
Kac-Moody
Algebras


Automorphic Products
Vertex Algebras
Generalized Kac-Moody Algebras

Definition

A **vertex operator algebra** (VOA) of *central charge* c consists of

- a $\mathbb{Z}_+$-graded complex vector space $V = \bigoplus_{m=0}^{\infty} V_m$ (*state space*) with $\dim V_m < \infty$ and $\dim V_0 = 1$;
- a vector $\mathbf{1} \in V_0$ (*vacuum vector*);
- a linear map

  $$Y(\,.\,,z) : V \longrightarrow \mathrm{End}(V)[[z,z^{-1}]] \qquad (\textit{vertex operator})$$

  which maps $a \in V_m$ to $Y(a,z) = \sum_{n\in\mathbb{Z}} z^{-n-1}$ such that a_n has degree $-n-1+m$ and for all $v \in V$ one has $a_n v = 0$ for large enough n.
- a vector $\omega \in V_2$ (*Virasoro vector*);

such that the following **axioms** are satisfied:

Definition

A **vertex operator algebra** (VOA) of *central charge* c consists of a tuple $(V, 1Y, \omega)$
such that the following **axioms** are satisfied:

- $Y(\mathbf{1}, z) = \mathrm{id}_V$, and for all $a \in V$ one has $Y(a, z)\mathbf{1} \in V[[z]]$ with $Y(a, z)\mathbf{1}|_{z=0} = a$ (*vacuum axiom*);
- For all a, $b \in V$ there exists an $N \in \mathbb{Z}_+$ such that

$$(z-w)^N Y(a, z) Y(b, w) = (z-w)^N Y(b, w)$$

  (*locality axiom*);
- for $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ one has (*Virasoro axioms*)
  - $L_0|_{V_m} = m \cdot \mathrm{id}|_{V_m}$,
  - $L_{-1}\mathbf{1} = 0$ and $[L_{-1}, Y(a, z)] = \partial_z Y(a, z)$ for all $a \in V$,
  - $L_2 \omega = \frac{c}{2} \cdot \mathbf{1}$.

Remark

The coefficients of $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ satisfy the commutator relations of the *Virasoro algebra:*

$$[L_m, L_n] = (n-m)L_{n+m} + \frac{n^3-n}{12}\,\delta_{n,-m}\cdot c\cdot \mathrm{id}_V.$$

Definition

The character of a VOA is the function

$$\chi_V(q) = \mathrm{tr}(q^{L_0-c/24}|V) = q^{-c/24}\sum_{n\in\mathbb{Z}_+} \dim\, V_n\cdot q^n.$$

Theorem (Reconstruction Theorem)

Let V be a $\mathbb{Z}_+$-graded vector space, $\mathbf{1} \in V_0$, $\omega \in V_2$ and $a^{(1)}, \ldots,$ $a^{(k)}$ be a set of homogenous vectors in V such that the corresponding vertex operators $Y(\mathbf{1}, z)$, $Y(\omega, z)$ and $Y(a^{(i)}, z) = \sum_{n\in\mathbb{Z}} a_n^{(i)} z^{-n-1}$, $i = 1, \ldots, k$, satisfy all the axioms of a VOA as far as defined and assume that the vectors $a_{j_1}^{(i_1)} \ldots a_{j_m}^{(i_m)}\mathbf{1}$, $j_\nu < 0$, generate V. Then the assignment

$$Y(a_{j_1}^{(i_1)} \ldots a_{j_m}^{(i_m)}\mathbf{1}, z)$$

$$= \frac{1}{(-j_1-1)!} \cdots \frac{1}{(-j_m-1)!} : \partial_z^{-j_1-1} Y(a^{(i_1)}, z) \ldots \partial_z^{-j_m-1} Y(a^{(i_m)}, z) :$$

give rise to a well-defined vertex algebra structure on V and is the unique one with the given $Y(a^{(i)}, z)$.

Example (Heisenberg VOA)

The graded polynomial ring $V_{\hat{h}} := \mathbb{C}[b_{-1}, b_{-2}, \ldots]$, $\deg b_n = -n$, has the structure of a vertex operator algebra of central charge 1 if we let

- $\mathbf{1} = 1$;
- $Y(b_{-1}, z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-1}$; where b_n acts for $n < 0$ by multiplication with b_n, for $n > 0$ acts as the derivation $n\frac{\partial}{\partial b_{-n}}$ and b_0 acts by 0, and define the remaining vertex operators using the reconstruction theorem;
- $\omega = \frac{1}{2} b_{-1}^2$.

Representation theory of VOAs

rational VOA $\longrightarrow$ 3d-TQFT (Witten 1989)

modular tensor category $\longrightarrow$ 3d-TQFTs (Reshetikin/Turaev)

rational VOA $\longrightarrow$ modular tensor category (Huang 2004)

Definition (Modules)

Let V be a VOA. A *V-module* is a graded vector space $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$ with $M_{\lambda+n} = 0$ for $n \in \mathbb{Z}$ small enough together with a map

$$Y_M(\, . \, , z) : V \longrightarrow \mathrm{End}(M)[[z, z^{-1}]]$$

which maps $a \in V_n$ to a field of conformal dimension n such that the following axioms holds:

- $Y_M(\mathbf{1}, z) = \mathrm{id}_M$
- $Y_M(a, z) Y_M(b, w) = Y_M(Y(a, z - w)b, w)$
- For $Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} L_n^M z^{-n-2}$ one has $L_0^M|_{M_\lambda} = \lambda \cdot \mathrm{id}_{M_0}$.

Definition

V *rational* $:\Longleftrightarrow$ every V-module is completely reducible.

A rational VOA (satisfying some extra conditions) has only finitely many isomorphism classes of irreducible modules.

Definition (Intertwining Operators)

Let M^1, M^2 and M^3 be three modules of a VOA V. A intertwineroperator of type $M^1 \times M^2 \longrightarrow M^3$ is a linear map

$$\mathcal{Y}(\,.\,,z) : M^1 \longrightarrow \mathrm{Hom}(M^2, M^3)\{z\},$$

satisfying certain natural axioms.
The intertwining space $\binom{M^3}{M^1\ M^2}$ is the space of all such maps.

Let $\{M^i\}_{i\in I}$ be the set of (isomorphism classes) of irreducible modules of a rational VOA V.
The vector space $A = \bigoplus_{i\in I} \mathbb{C}[M^i]$ with product defined by

$$[M^i] \times [M^j] = \sum_{k\in I} \dim \binom{M^k}{M^i\ M^j} [M^k]$$

is a commutative and associative algebra called the *fusion algebra*.

Remark

The Heisenberg VOA $V_{\hat{h}}$ is not rational, but has infinitely many irreducible modules parameterized by $\mathbb{C}$.
Fusion algebra $A \cong \mathbb{C}[\mathbb{C}]$.

In general

$$\mathcal{Y}(\,.\,,z)\mathcal{Y}'(\,.\,,w) \neq \mathcal{Y}'(\,.\,,w)\mathcal{Y}(\,.\,,z)$$

$\longrightarrow$ braiding map.

Theorem (Huang)

The module category of a rational VOA V (satisfying some extra conditions) forms a modular (braided tensor) category $\mathcal{T}(V)$.

This talk: interested in very special case $A = \mathbb{C}[I] \cong \mathbb{C}[G]$ for a finite abelian group G ("abelian fusion"). Then $\mathcal{T}(V)$ is determined by G and the quadratic form $q : G \longrightarrow \mathbb{Q}/\mathbb{Z}$, defined by associating an irreducible module M^i the number $h_i \pmod{\mathbb{Z}}$, where h_i is the *conformal weight* (the smallest λ with $M^i_\lambda \neq 0$).

Examples of rational VOAs with abelian fusion

Example (Lattice VOAs)

Let $L \subset \mathbb{R}^n$ be a positive definite *even integral lattice.*
Let $L' = \{x \in \mathbb{R}^n \mid (x, y) \in \mathbb{Z} \text{ for all } y \in L\} \supset L$ be the dual lattice.
Then $V_L := V_{\hat{h}}^{\otimes n} \otimes \mathbb{C}[L]$ has the structure of a rational VOA with fusion algebra $\mathbb{C}[L'/L]$ and the quadratic form $q : L'/L \longrightarrow \mathbb{Q}/\mathbb{Z}$ is the discriminant form $x \mapsto (x, x)/2 \pmod{\mathbb{Z}}$.

Example (A toy example (Griess 1998))

The *automorphism group* of a VOA V is defined by

$$\mathrm{Aut}(V) = \{g \in \mathrm{End}(V) \mid g\mathbf{1} = 1,\ g\omega = \omega,\ Y(g\,.\,,z)g = gY(\,.\,,z)\}.$$

For $G < \mathrm{Aut}(V)$ the *fixpoint VOA* is defined by

$$V^G = \{v \in V \mid gv = v \text{ for all } g \in G\}.$$

Example:

For L the root lattice E_8 one has $\mathrm{Aut}(V_{E_8}) = E_8(\mathbb{C})$.
Let $H \cong (\mathbb{Z}/2\mathbb{Z})^5 < E_8(\mathbb{C})$ be the $2B$-pure subgroup of rank 5.
The VOA $V_{E_8}^H$ is a rational VOA with fusion algebra $A \cong \mathbb{C}[(\mathbb{Z}/2\mathbb{Z})^{10}]$ and quadratic form $q : (\mathbb{Z}/2\mathbb{Z})^{10} \longrightarrow \mathbb{Q}/\mathbb{Z}$ such that $O((\mathbb{Z}/2\mathbb{Z})^{10}, q) = O^+(10, 2)$.
Furthermore, the natural map $\mathrm{Aut}(V_{E_8}^H) \longrightarrow O((\mathbb{Z}/2\mathbb{Z})^{10}, q)$ is an isomorphism.

Example (Simple orbifolds)

Let V be a *self-dual* (also called holomorphic) VOA: A rational VOA with only one irreducible module, namely V itself.
Then $A = \mathbb{C}[\{e\}] \cong \mathbb{C}$ and $q : \{e\} \longrightarrow \mathbb{Q}/\mathbb{Z}$ is the zero map.
Let $g \in \mathrm{Aut}(V)$ be of finite order n. Then (conjecturally) $V^{\langle g \rangle}$ is a rational VOA with fusion algebra $A \cong \mathbb{C}[\mathbb{Z}/hn\mathbb{Z} \times \mathbb{Z}/(n/h)\mathbb{Z}]$ for some $h|n$.

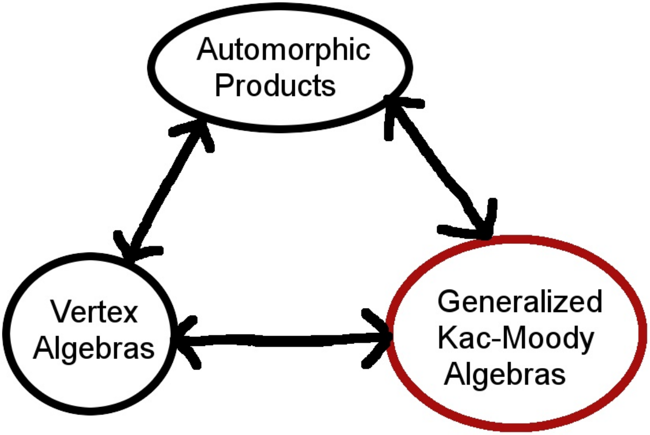
Example: Let $\Lambda \subset \mathbb{R}^{24}$ be the Leech lattice, the densest lattice packing in dimension 24. Since $\Lambda = \Lambda'$, the lattice VOA V_Λ is self-dual. Let $\tau : V_\Lambda \longrightarrow V_\Lambda$ be the (well-defined) lift of order two of the map $-1 : \Lambda \longrightarrow \Lambda$ to V_Λ. Then $V_\Lambda^+ := V_\Lambda^{\langle \tau \rangle}$ has fusion algebra $A = \mathbb{C}[\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}]$ and the four modules V_Λ^+, V_Λ^-, $(V_\Lambda^T)^+$ and $(V_\Lambda^T)^-$ have the conformal weights 0, 1, 2 and 3/2, respectively (Huang, Abe/Dong/Li).

Example (Extensions)

Let V be a rational VOA with fusion algebra $A \cong \mathbb{C}[G]$ and quadratic form $q : G \longrightarrow \mathbb{Q}/\mathbb{Z}$.
Let $C \subset G$ be a totally isotropic subgroup of the finite quadratic space (G, q). Then $W := \bigoplus_{c \in C} M(c)$ has the structure of a rational VOA with fusion algebra $A \cong \mathbb{C}[C^*/C]$ and quadratic form $C^*/C \longrightarrow \mathbb{Q}/\mathbb{Z}$ the one induced by q.

Example: Let $V = V_\Lambda^+$ as in the previous example. Since $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the modules of V_Λ^+ have conformal weights 0, 1, 2 and 3/2 there are two isotropic subgroups $C \cong \mathbb{Z}/2\mathbb{Z}$ leading to the VOAs $W = V_\Lambda^+ \oplus V_\Lambda^- \cong V_\Lambda$ and $W' = V_\Lambda^+ \oplus (V_\Lambda^T)^- = V^\natural$, the *moonshine module.*
One has $\mathrm{Aut}(V^\natural) = \mathbf{M}$, the monster — the largest sporadic simple group of order $2^{46}3^{20}5^9 7^6 11^2 13^3 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$ (…, Fischer, Griess, Tits, Borcherds, Frenkel / Lepowsky / Meurmann).

We also have to consider indefinite lattices L. Then $V = V_L$ is still defined but $V_m \neq 0$ for $m < 0$ and $\dim V_m = \infty$ may happen. This leads to the more general notation of **vertex algebras**.


Automorphic Products
Vertex Algebras
Generalized Kac-Moody Algebras

Let $A = (a_{ij})_{i,j \in I}$ be a real symmetric square matrix for some finite or countable index set I with the properties

$$\begin{aligned} a_{ij} \le 0 &\quad \text{if} \quad i \ne j, \\ 2a_{ij}/a_{ii} \in \mathbb{Z} &\quad \text{if} \quad a_{ii} > 0. \end{aligned}$$

Let $\hat{\mathbf{g}}$ be the real or complex Lie algebra with generators e_i, f_i, h_{ij} for i, $j \in I$ and relations

$$\begin{aligned} [e_i, f_j] &= h_{ij}, \\ [h_{ij}, e_k] &= \delta_i^j a_{ik}\, e_k, \\ [h_{ij}, f_k] &= -\delta_i^j a_{ik}\, f_k, \\ (\mathrm{ad}\, e_i)^{1-2a_{ij}/a_{ii}} e_j &= (\mathrm{ad}\, f_i)^{1-2a_{ij}/a_{ii}} f_j \;=\; 0, \\ [e_i, e_j] &= [f_i, f_j] \;=\; 0 \;\text{ if } a_{ij} = 0. \end{aligned}$$

A Lie algebra $\mathbf{g}$ is called a *generalized Kac-Moody algebra* if $\mathbf{g}$ is the semi-direct product $(\hat{\mathbf{g}}/C) \cdot D$ where C is a subspace of the center of $\hat{\mathbf{g}}$ and D is an abelian subalgebra of $\mathbf{g}$ such that the elements e_i and f_i are all eigenvectors for D.

Let **g** be a generalized Kac-Moody algebra. We keep the notion e_i, f_i, h_i for the images in the quotient.

The **root lattice** Q of **g** is the free abelian group generated by elements α_i, $i \in I$, and the bilinear form defined by $(\alpha_i, \alpha_j) = a_{ij}$. The elements α_i are called simple roots. $\hat{\mathbf{g}}$ is graded by Q. A root is called **real** if it has positive norm and **imaginary** else. The space $H = (\hat{H}/C) \oplus D$ is a commutative subalgebra of **g** called the **Cartan subalgebra**.

There is a unique invariant symmetric bilinear form on **g** satisfying $(h_i, h_j) = a_{ij}$. We have a natural homomorphism of abelian groups from Q to H sending α_i to h_i. By abuse of terminology the images of roots under this map are also called roots. It is important to note that this map is in general not injective. It is possible that $n > 1$ imaginary simple roots have the same image h in H. In this case we call h a root with multiplicity n.

A vector $\rho \in H$ is called a Weyl vector if one has $(\rho, \alpha) = -(\alpha, \alpha)/2$ for all simple roots α.

Theorem (Borcherds-Kac-Weyl Character formula)

Let V_λ be the irreducible highest weight module of integrable highest weight λ for a generalized Kac-Moody algebra $\mathbf{g}$. *Then*

$$\chi(V_\lambda) = \frac{\sum_{w\in W} \operatorname{sgn}(w) w\left(\sum_S (-1)^{|S|} e^{\rho+\lambda+\sum_{\alpha\in S}\alpha}\right)}{e^\rho \prod_{\alpha>0}(1-e^{-\alpha})^{\operatorname{mult}\alpha}}$$

where W is the Weyl group generated by the reflections at the real roots, ρ is a Weyl vector, S runs through all sets of mutually orthogonal imaginary simple roots and the product in the denominator is over all positive roots.

In the special case $\lambda = 0$ this gives the **denominator identity** of $\mathbf{g}$:

$$\sum_{w\in W} \operatorname{sgn}(w) w\left(\sum_S (-1)^{|S|} e^{\rho+\sum_{\alpha\in S}\alpha}\right) = e^\rho \prod_{\alpha>0}(1-e^{-\alpha})^{\operatorname{mult}\alpha}.$$

Finding generalized Kac-Moody algebras

Theorem (Borcherds 1995)

A Lie algebra $\mathbf{g}$ satisfying the following five conditions is a generalized Kac-Moody algebra

i) $\mathbf{g}$ *has a nonsingular invariant symmetric bilinear form* $(\,,\,)$.

ii) $\mathbf{g}$ *has a self centralizing subalgebra* H *such that* $\mathbf{g}$ *is the sum of eigenspaces of* H *and all eigenspaces are finite dimensional.*

iii) *The bilinear form restricted to* $Q \otimes \mathbf{R} \subset H$ *is Lorentzian.*

iv) *The norms of roots of* $\mathbf{g}$ *are bounded above.*

v) *If two roots are positive multiples of the same norm zero vector then their root spaces commute.*

Problem: Find *interesting examples*:

- Those which have a natural construction.
- Those for which one can describe the simple roots.

BRST-cohomology construction from string theory

Certain vertex algebras $W \longrightarrow$ generalized Kac-Moody algebras $\mathbf{g}$

There is an action of the BRST-operator on the tensor product of a vertex algebra W of central charge 26 with the bosonic ghost vertex superalgebra V_{ghost} of central charge -26, which defines the BRST-cohomology groups $H^{*}_{\mathrm{BRST}}(W)$.

The degree 1 cohomology group $H^{1}_{\mathrm{BRST}}(W)$ has additionally the structure of a Lie algebra (Frenkel/Garland/Zuckerman 1986).

Let W be the vertex algebra of central charge 26 such that:

- $V_L \subset W$ for some even integral Lorentzian lattice L, of rank $k \geq 2$.
- $U := \mathrm{Com}_W(V_L)$ is a VOA of central charge $26 - k$ such that $U_1 = 0$.

— V_L-module decomposition: $W = \bigoplus_{\gamma \in L'/L} U(\gamma) \otimes V_{L+\gamma}$
— induced L'-grading: $W = \bigoplus_{\alpha \in L} W(\alpha)$.

The **bosonic ghost vertex superalgebra** V_{ghost} is defined as the vertex superalgebra for the integer lattice $\mathbb{Z}\sigma$, $(\sigma, \sigma) = 1$, together with the Virasoro element

$$\omega^{\mathrm{Gh}} = \tfrac{1}{2}\sigma_{-1}^2 + \tfrac{3}{2}\sigma_{-2}$$

of central charge -26. The states

$$b := e^{-\sigma} \qquad \text{and} \qquad c := e^{\sigma}$$

have L_0^{Gh} eigenvalues 2 and -1, respectively.

Inside $W \otimes V_{\mathrm{ghost}}$

- The Virasoro element

$$\omega = \omega^W + \omega^{\mathrm{Gh}}$$

  of central charge 0.

- The BRST-current

$$j^{\mathrm{BRST}} = c_{-1}\omega^W + \tfrac{1}{2}c_{-1}\omega^{\mathrm{Gh}}.$$

  The zero mode $Q := j_0^{\mathrm{BRST}}$ is called the **BRST-operator**.

- Let

$$j^N = \sigma_{-1}.$$

  The zero mode $\overline{U} := j_0^N$ is called the **ghost number operator**.

The operators $\overline{U}$ and L_0 are commuting, act diagonally and respect the L'-grading of $W = \bigoplus_{\alpha \in L'} W(\alpha)$.
Let $W(\alpha)^u_n$ be the subspace of $W(\alpha)$ on which $\overline{U}$ and L_0 act by the eigenvalues u and n.
The equations

$$Q^2 = 0, \qquad [\overline{U}, Q] = Q, \qquad [Q, L_0] = 0$$

show that one has a complex

$$\cdots \xrightarrow{Q} W(\alpha)^{u-1}_n \xrightarrow{Q} W(\alpha)^u_n \xrightarrow{Q} W(\alpha)^{u+1}_n \xrightarrow{Q} \cdots$$

leading to cohomology groups $H^*_{\mathrm{BRST}}(W) = \bigoplus_{\alpha,\, n} H^*(\alpha)_n$.

Theorem

On $H^1_{\mathrm{BRST}}(W)$ there is the structure of a Lie algebra $\mathbf{g}$, the Lie algebra $\mathbf{g}$ is graded by the lattice L' and its components $\mathbf{g}(\alpha)$ for $\alpha \in L'$ are isomorphic to $(U([\alpha]) \otimes V_{\hat{h}}^{\otimes(k-2)})_{1-\alpha^2/2}$ for $\alpha \neq 0$ and to $\mathbf{R}^k$ for $\alpha = 0$.

Proof: For the Lie algebra structure, cf. Lian/Zuckerman 1989. For the identification of the components $\mathbf{g}(\alpha) = \bigoplus_n H^1(\alpha)_n$, one defines the relative BRST-complex: Let $B(\alpha)^u_n = W(\alpha)^u_n \cap \ker b_1$ and $C(\alpha)^u = B(\alpha)^u_0$. From equation $[Q, b_1] = L_0$ we get the subcomplex

$$\cdots \xrightarrow{Q} C(\alpha)^{u-1} \xrightarrow{Q} C(\alpha)^u \xrightarrow{Q} C(\alpha)^{u+1} \xrightarrow{Q} \cdots$$

leading to the relative cohomology groups $H^*_{\mathrm{rel}}(C) = \bigoplus_{\alpha,\, n} H^*_{\mathrm{rel}}(\alpha)$. The BRST- and the relative BRST-complex are connected by a short exact sequence of cochain complexes leading to the isomorphism $H^1_{\mathrm{BRST}}(W) = H^0_{\mathrm{rel}}(C)$.

The relative BRST-cohomology groups can be identified with the relative semi-infinite cohomology groups for the Virasoro algebra as introduced by Feigin:

$$H^u_{\mathrm{rel}}(\alpha) = H^u_{\infty/2}(\mathrm{Vir}, \mathrm{Vir}_0; U([\alpha]) \otimes V_{\hat{h}}^{\otimes k}(\alpha)),$$

where $V_{\hat{h}}^{\otimes k}(\alpha)$ is the highest weight module of highest weight α for the rank k Heisenberg vertex algebra of Lorentzian signature. The vanishing theorem of Feigin gives for $\alpha \neq 0$:

$$H^u_{\infty/2}(\mathrm{Vir}, \mathrm{Vir}_0; U([\alpha]) \otimes V_{\hat{h}}^{\otimes k}(\alpha)) = 0 \qquad \text{for } u \neq 0.$$

It follows that $H^0_{\mathrm{rel}}(\alpha)$ can be determined from the $B(\alpha)^u_n$ by the Euler-Poincare principle and counting the states of conformal weight $n = 0$. One finds

$$\dim H^0_{\mathrm{rel}}(\alpha) = \text{Coefficient of } q^0 \text{ in } q^{(\alpha,\alpha)/2} \chi_{U([\alpha]) \otimes V_{\hat{h}}^{\otimes (k-2)}}(q).$$

For $\alpha = 0$ one finds $H^0_{\mathrm{rel}}(0) \cong \mathbf{R}^k$. **Q.E.D.**

Theorem

The Lie algebra $\mathbf{g}$ *is a generalized Kac-Moody algebra.*

Examples

Example (The fake monster Lie algebra)

Let $II_{1,1}$ be the unique even unimodular Lorentzian lattice of rank 2. Let $W := V_\Lambda \otimes V_{II_{1,1}} \cong V_{\Lambda \oplus II_{1,1}} \cong V_{II_{25,1}}$, where $II_{25,1}$ is the unique even unimodular Lorentzian lattice of rank 26.

The $II_{25,1}$-graded generalized Kac-Moody Lie algebra $\mathbf{g}^\Lambda := H^1_{\mathrm{BRST}}(W)$ is called the **fake monster lie algebra** (Borcherds 1989).

Example (The monster Lie algebra)

Let $W := V^\natural \otimes V_{II_{1,1}}$, where $V^\natural$ is the Moonshine module.

The $II_{1,1}$-graded generalized Kac-Moody Lie algebra $\mathbf{g}^\natural := H^1_{\mathrm{BRST}}(W)$ is called the **monster Lie algebra** (Borcherds 1992).

Example (The fake baby monster Lie algebra)

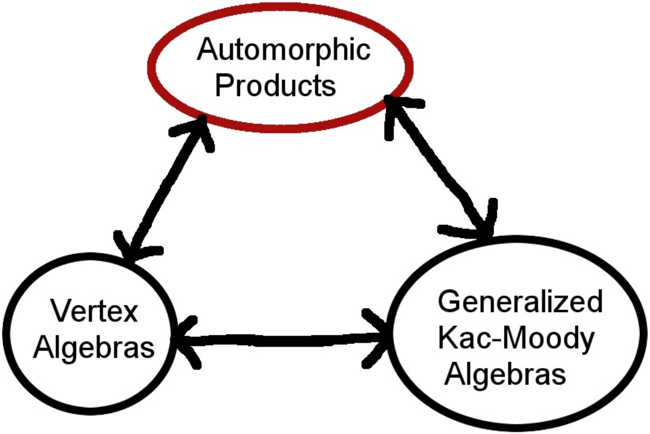
Let N be the even unimodular lattice of rank 24 with root lattice A_3^8. Let $\widetilde{V}_N$ be the $\mathbb{Z}_2$-orbifold of V_N (using the same construction as for $V^\natural$ from V_Λ). Let $W := \widetilde{V}_N \otimes V_{II_{1,1}}$.
One has $W \cong \bigoplus_{\gamma \in L'/L} V_{E_8}^H(\gamma) \otimes V_{L+\gamma}$, where $L \cong BW_{16} \oplus II_{1,1}(2)$ is the unique even Lorentzian lattice of rank 18 with discriminant form $(G, -q)$. Here, (G, q) is the quadratic space determined by the tensor category of the "toy example" $V_{E_8}^H$.

The L'-graded generalized Kac-Moody Lie algebra $\mathbf{g}_{BW}^\Lambda := H^1_{\mathrm{BRST}}(W)$ is called the **fake baby monster Lie algebra** (Höhn/Scheithauer 2002).

Example (The baby monster Lie algebra)

Let $II_{1,1}(2)$ be the even Lorentzian lattice of rank 2 obtained from $II_{1,1}$ by scaling the norms by a factor 2. Let t be a $2A$-involution of the monster group. The fixed point VOA $(V^\natural)^{\langle t \rangle}$ has fusion algebra $\mathbb{C}[\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}]$ and its modules can be labeled by $II_{1,1}(2)'/II_{1,1}(2)$.
Let $W := \bigoplus_{\gamma \in II_{1,1}(2)'/II_{1,1}(2)} (V^\natural)^{\langle t \rangle}(\gamma) \otimes V_{II_{1,1}(2)+\gamma}$.

The $II_{1,1}(2)'$-graded generalized Kac-Moody Lie algebra $\mathbf{g}^\natural_{2A} := H^1_{\mathrm{BRST}}(W)$ is called the **baby monster Lie algebra** (H. 2003).


Automorphic Products
Vertex Algebras
Generalized Kac-Moody Algebras

The Borcherds lift

VOAs $\leftrightarrow$ automorphic products $\leftrightarrow$ generalized Kac-Moody algebras

To show that the grading lattices L' of the last four examples are indeed the root lattices and to determine the simple roots one has to discuss the singular theta-correspondence as developed by Borcherds.

Let L be an even lattice of signature $(k-1,1)$ and denote with L' the dual lattice of L.

The discriminant form L'/L defines a representation ρ_L of $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ (the *Weil representation*) on the group ring $\mathbb{C}[L'/L]$ by

$$\begin{aligned} \rho_L(T)\, e^\gamma &= e^{-\pi i(\gamma,\gamma)}\, e^\gamma, \\ \rho_L(S)\, e^\gamma &= \frac{e^{2\pi i\cdot\mathrm{sign}(L)/8}}{\sqrt{|L'/L|}} \sum_{\beta\in L'/L} e^{2\pi i(\gamma,\beta)}\, e^\beta, \end{aligned}$$

where $S = \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right)$ and $T = \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right)$ are the standard generators of $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$.

Let m be in $\frac{1}{2}\mathbb{Z}$. A modular form for $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ of weight m and the Weil representation ρ_L is a holomorphic map F from the upper half plane to $\mathbb{C}[L'/L]$ such that

$$F\left(\frac{a\tau+b}{c\tau+d}\right) = (\pm\sqrt{c\tau+d}\,)^{2m}\rho_L(g)F(\tau)$$

for all $g = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \pm\sqrt{c\tau+d}\right)$ in $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$. Singularities at the cups are allowed.

Theorem (Borcherds 1998)

Let $M = L \oplus II_{1,1}$ be an even lattice of signature $(k, 2)$ and $F = (f^\gamma)_{\gamma \in L'/L}$ be a modular form of weight $1 - k/2$ and representation ρ_L which is holomorphic on the upper half plane and meromorphic at cusps and whose Fourier coefficients are integers for nonpositive exponents. Then there is a meromorphic function $\Psi_L(Z_L, F)$ (the Borcherds lift) with the following properties:

1. *$\Psi_L(Z_L, F)$ is an automorphic form of weight given by the constant term of f^0 divided by 2 on the Grassmannian $\mathrm{Gr}_2(M \otimes \mathbf{R})$.*
2. *If in addition, L has dimension at least 3 and the constant term of f^0 is $b^+ - 2$, then Ψ_L is a holomorphic automorphic form, has singular weight and the only nonzero Fourier coefficients of Ψ_L correspond to norm 0 vectors in L.*

We can apply the Borcherds lift to the generalized Kac-Moody algebra $\mathbf{g}$ obtained from W if we assume that in the V_L-module decomposition

$$W = \bigoplus_{\gamma \in L'/L} U(\gamma) \otimes V_{L+\gamma}$$

that U has *abelian fusion* and the $U(\gamma)$ run through the irreducible U-modules. In this case W is a self-dual vertex algebra.

The function $F = (f^\gamma/\eta^{k-2})_{\gamma \in L'/L}$ where $f^\gamma = \chi_{U(\gamma)}$ is now a vector valued modular function of the required type for the Weil representation for L'/L. This follows from Zhu's result about the modular invariance of the character of the modules of a VOA together with results about the tensor category $\mathcal{T}(U)$

The Borcherds lift $\Psi_L(Z_L, F)$ can now be interpreted as the product side of the denominator identity of $\mathbf{g}$. The Fourier expansion can be interpreted as the other side, allowing to identify the root lattice, the simple roots and the Weylgroup of $\mathbf{g}$.

Example (The fake monster Lie algebra $\mathbf{g}_{BW}^{\Lambda}$)

- *root lattice:* $L = \Lambda \oplus II_{1,1}$ with elements $(\mathbf{s}, m, n)$, $s \in \Lambda$, m, $n \in \mathbb{Z}$ and norm $(\mathbf{s}, m, n)^2 = \mathbf{s}^2 - 2mn$,
- *Weyl vector:* $\rho = (\mathbf{0}, 0, 1)$,
- *real simple roots:* the norm 2 vectors $(\mathbf{s}, 1, (\mathbf{s}^2 - 2)/2) \in L$,
- *imaginary simple roots:* $n\rho$, $n \in \mathbb{Z}_+$, with multiplicity 24.
- *denominator identity:*

$$e^{\rho} \prod_{\alpha \in L^+} (1 - e^{\alpha})^{c(-\alpha^2/2)}$$

$$= \sum_{w \in W} \det(w)\, w \left(e^{\rho} \prod_{n=1}^{\infty} (1 - e^{n\rho})^{24} \right)$$

where $c(n)$ is the coefficient of q^n in

$$\eta(\tau)^{-24} = q^{-1} + 24 + 324\, q + 3200\, q^2 + 25650\, q^3 + 176256\, q^4 + \cdots .$$

Example (The monster Lie algebra $\mathbf{g}^\natural$)

- *root lattice:* $L = II_{1,1}$ with elements (m,n), m, $n \in \mathbb{Z}$ and norm $(m,n)^2 = -2mn$,
- *Weyl vector:* $\rho = (-1,0)$,
- *real simple roots:* $(1,-1)$,
- *imaginary simple roots:* $(1,n)$ with multiplicity $c(n)$ for $n \in \mathbb{Z}$,
- *denominator identity:*

$$p^{-1} \prod_{\substack{m\in\mathbb{Z},\, m>0 \\ n\in\mathbb{Z}}} (1 - p^m q^n)^{c(mn)} = j(p) - j(q)$$

where $p = e^{(1,0)}$, $q = e^{(0,1)}$, and $c(n)$ is the coefficient of q^n in

$$\chi_{V^\natural} = j(q) - 744 = q^{-1} + 196844\, q + 21493769\, q^2 + 864299970\, q^3 + \cdots$$

Example (The fake baby monster Lie algebra $\mathbf{g}^{\Lambda}_{BW}$)

- *root lattice:* L', where $L = BW_{16} \oplus II_{1,1}(2)$ with elements $(\mathbf{s}, m, n)$, $\mathbf{s} \in BW_{16}$, m, $n \in \mathbb{Z}$ and norm $(\mathbf{s}, m, n)^2 = \mathbf{s}^2 - 4mn$,
- *Weyl vector:* $\rho = (\mathbf{0}, 0, 1/2)$,
- *real simple roots:* norm 1 vectors in L' of the form $(\mathbf{s}, 1/2, (\mathbf{s}^2 - 1)/2)$, $\mathbf{s} \in BW'_{16}$, and the norm 2 vectors $(\mathbf{s}, 1, (\mathbf{s}^2 - 2)/4)$ in L,
- *imaginary simple roots:* $n\rho$, $n \in \mathbb{Z}_+$, with multiplicity $\begin{cases} 16, & \text{for even } n, \\ 8, & \text{for odd } n. \end{cases}$
- *denominator identity:*

$$e^{\rho} \prod_{\alpha \in L^+} (1 - e^{\alpha})^{c(-\alpha^2/2)} \prod_{\alpha \in L'^+} (1 - e^{\alpha})^{c(-\alpha^2)}$$

$$= \sum_{w \in W} \det(w)\, w \left(e^{\rho} \prod_{n=1}^{\infty} (1 - e^{n\rho})^8 (1 - e^{2n\rho})^8 \right)$$

where W is the reflection group generated by norm 1 vectors of L' and the norm 2 vectors of L and $c(n)$ is the coefficient of q^n in

$$\eta(\tau)^{-8}\eta(2\tau)^{-8} = q^{-1} + 8 + 52\,q + 256\,q^2 + 1122\,q^3 + 4352\,q^4 + \cdots .$$

Example (The baby monster Lie algebra $\mathbf{g}^\natural_{2A}$)

- *root lattice:* $L' = II_{1,1}(2)'$ with elements (m,n), $m,\ n \in \frac{1}{2}\mathbb{Z}$ & norm $(m,n)^2 = -4mn$,
- *Weyl vector:* $\rho = (-\frac{1}{2}, 0)$,
- *real simple roots:* $(\frac{1}{2}, -\frac{1}{2})$,
- *imaginary simple roots:* $(\frac{1}{2}, n)$ with multiplicity $\begin{cases} c^{10}(n), & \text{for } n \in \mathbb{Z}, \\ c^{11}(n), & \text{for } n \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}, \end{cases}$
- *denominator identity:*

$$p^{-1/2} \prod_{\substack{m \in \frac{1}{2}\mathbb{Z},\ m>0 \\ n \in \frac{1}{2}\mathbb{Z}}} (1 - p^m q^n)^{c^{[m,n]}(2mn)} = f(p) - f(q)$$

where $p = e^{(1,0)}$, $q = e^{(0,1)}$, $[m,n]$ denotes the rest class of $(m,n) \in L'$ in L'/L, $c^{[ij]}(n)$ is the coefficient of q^n of the character $f^{ij}(q)$ of the $(V^\natural)^{\langle t \rangle}$-module $(V^\natural)^{\langle t \rangle}([ij])$ and

$$f(q) = f^{10}(q) + f^{11}(q) = q^{-1/2} + 95256\, q^{1/2} + \cdots .$$

Classification of generalized Kac-Moody algebras

- The described construction works for *self-dual* vertex algebras W containing a Lorentzian lattice L of rank ≥ 2.
- Equivalent to classify those W seems to be to classify the vertex operator algebras U of central charge ≤ 24 with $U_1 = 0$ and abelian fusion since there seems to be always a unique "complementary lattice" L.
- It seems that for $\mathrm{rank}\ L > 2$ one has a decomposition $L \cong K \oplus II_{1,1}(d)$ such that the genus of K is always "quasi-reflective" in the sense that there is exactly one lattice without roots in the genus and the other lattices have root lattices of maximal rank. There may exist a lattice theoretic classification of such L.
- The functions $(f^\gamma/\eta^{k-2})_{\gamma \in L'/L}$ define a vector valued modular function for the Weil representation of the discriminant group L'/L. One can try to classify those (Barnard, Scheithauer ($\mathrm{rank}\ L > 2$), Cummins ($\mathrm{rank}\ L = 2$)).
- Nicer would be a characterization of the automorphic products corresponding to such Borcherds lifts.

There are two (related) ways to obtain new generalized Kac-Moody (super)algebras from $\mathbf{g}$ for an automorphism $g \in \mathrm{Aut}(W)$. One either replaces U by the $\widetilde{U}$ in $W^{\langle g \rangle}$ and obtains a new generalized Kac-Moody algebra $\mathbf{g}_g$ from $\widetilde{U}$ or one considers the g-equivariant denominator identity of $\mathbf{g}$ and interprets it as the denominator identity of a new generalized Kac-Moody (super)algebra $\widetilde{\mathbf{g}}_g$. Borcherds conjectured that all good examples of generalized Kac-Moody algebras can be obtained in this way by either "twisting" the *fake monster* or the *monster* Lie algebra. These examples have been worked out by Scheithauer and Carnahan, respectively.

Problem: Are there further examples arising in the way as described in the talk?

Generalized Kac-Moody Lie superalgebras

A vertex algebra construction of the fake monster superalgebra has been given by Scheithauer (2000). This Lie algebra arises from the super string construction and has a ten-dimensional root lattice and no real roots. Scheithauer also considered twists of this Lie superalgebra.

Sometimes, the expansion of the denominator identity of a generalized Kac-Moody Lie algebra in another cusp gives the denominator identity of a generalized Kac-Moody Lie superalgebra.

Problem: Generalize Scheithauer's vertex operator theoretic construction of the fake monster superalgebra to those other Lie superalgebras.

Monstrous Moonshine

It was conjectured by Conway and Norton that the *McKay-Thompson series*

$$T_g = q^{-1} \sum_{n \in \mathbb{Z}_+} \mathrm{tr}(g|V^\natural) q^n$$

for $g \in \mathbf{M} = \mathrm{Aut}(V^\natural)$ should be explicitily given modular functions for genus zero groups.

The T_g have been determined by Borcherds. The equivariant denominator identity of the monster Lie algebra

$$\mathrm{tr}(g|\Lambda^*(E)) = \mathrm{tr}(g|H^*(E))$$

implies that they are *completely replicable*. A series

$$q^{-1} + H_1\, q + H_2\, q^2 + H_3\, q^3 + \cdots$$

is called completely replicable if certain recursion equations are satisfied. Completely replicable functions are determined by the coefficients H_1, H_2, H_3 and H_5.

On the other hand, the modular functions explicitily given by Conway and Norton are also known to be completely replicable. Since the monster module structure of $V_n^\natural$ is known for $n \leq 6$ it can be directly checked that the T_g are indeed the modular functions predicted by Conway and Norton.

Cummins and Gannon have shown that completely replicable functions are modular functions for genus zero groups without case-by-case checking.

The modular functions T_g are the Hauptmoduls for groups $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ which have been characterized as follows (Conway, McKay and Sebbar 2004):

- they are of type $n|h + e_1, \ldots, e_l$,
- $\mathbf{H}/\Gamma$ has genus zero,
- Γ is an extension of $\Gamma_0(nh)$ by a group of exponent 2,
- Each cusp of Γ can be mapped to the cusp ∞ by an element $\sigma \in \mathrm{PSL}_2(\mathbb{R})$ such that $\sigma\Gamma\sigma^{-1}$ has width 1 at ∞ and the intersection $\Gamma \cap \sigma\Gamma\sigma^{-1}$ contains $\Gamma_0(nh)$.

There are 171 such functions. On the other hand, the T_g depend only on the conjugacy class of g in the Monster and $T_g = T_{g^{-1}}$. There are 194 different conjugacy classes in the Monster and 172 pairs $\{[g], [g^{-1}]\}$. The two classes of type 27A and 27B have the same McKay-Thompson series but are not inverse to each other.

Since $V_1^\natural = 0$, the moonshine module is an example of an *extremal VOA* (Höhn 1995). Witten conjectured in 2007 that extremal VOAs exist for all central charges $c = 24k$, are unique and describe pure three-dimensional quantum gravity. Recently, Duncan and I. Frenkel have reformulated the above characterization of the T_g in terms of Rademacher sums and discussed possible connections with three-dimensional quantum gravity.

Norton conjectured that the *generalized McKay-Thompson series*

$$T_{g,h} = q^{-1} \sum_{n \in \mathbb{Z}_+} \mathrm{tr}(h|V^\natural(g))\, q^n$$

for $h \in C_{\mathbf{M}}(g)$ are also explicitily given modular functions for genus zero groups.
For $g = t$ an $2A$-involution of the monster, I determined the $T_{t,h}$ by using the equivariant denominator of the baby monster Lie algebra and showing that they are *replicable.*
Replicable functions are determined by their first 25 coefficients.
The modular functions explicitily given by Norton in this case are also replicable. Using the baby monster module structure of $VB^\natural_n$ for $n \leq 24$, I could check directly that the $T_{t,h}$ are the modular functions found by Norton.

The approach has been generalized by Carnahan to all conjugacy classes g of the monster using the Lie algebras $\mathbf{g}^\natural_g$. He proves the genus zero property for $T_{t,h}$ for roughly half of the g by a similar argument as used by Cummins and Gannon for completely replicable functions and gives no explicit identification of the $T_{t,h}$.

Possible application: Elliptic genera

The index of the Dirac operator on a Spin-manifold induces a ring homomorphism

$$\hat{A} : \Omega_*^{\mathrm{Spin}} \longrightarrow \mathbb{Z}.$$

The formal S^1-equivariant index of the Dirac operator of the loop space $\mathcal{L}X := \{\alpha : S^1 \longrightarrow X\}$ of a String-manifold X induces a ring homomorphism

$$\hat{A}(q, \mathcal{L}X) : \Omega_*^{\mathrm{String}} \longrightarrow \mathbb{Z}[E_4/\eta^8, E_6/\eta^{12}]$$

called the *Witten genus*.

Hirzebruch asked if there exists a 24-dimensional string manifold $X^\natural$ on which the monster acts in such a way that for the equivariant Witten genus one has

$$\hat{A}(g; q, \mathcal{L}X^\natural) = T_g.$$

Hopkins and Mahowald showed that there exists a string manifold X with $\hat{A}(q, \mathcal{L}X) = j - 744 = T_e$, but Hirzebruch's question seems still be open.

For a space X let $\exp(pX) := \sum_{n=0}^{\infty} X^n/S_n \cdot p^n$ be the generating series of its symmetric powers. The Witten genus of an orbifold X/G for a finite group G is defined by

$$\hat{A}(q, \mathcal{L}(X/G)) := \frac{1}{|G|} \sum_{[g,h]=1} \hat{A}(g; q, \mathcal{L}_h X).$$

It follows from calculations similar as done by Verlinde, Verlinde, Dijkgraaf and Moore that the 2nd quantized elliptic genus $\hat{A}(q, \mathcal{L}\exp(pX))$ is up to an automorphic correction factor the Borcherds lift of $\hat{A}(q, \mathcal{L}X)$.
Assuming the existence of a monster manifold $X^\natural$, there is an induced monster action on $\exp(pX^\natural)$. A similar calculation as before shows that for the monster equivariant 2nd quantized elliptic genus one gets

$$\hat{A}(g; q, \mathcal{L}\exp(pX)) = \mathrm{tr}(g|\Lambda^*(E)),$$

i.e. one side of the equivariant denominator identity of the monster Lie algebra.

Problem: Is there any topological interpretation of the monster Lie algebra in terms of the monster manifold $X^\natural$ provided it exists?

Possible application: $K3$ surfaces

- The Picard lattices of $K3$ surfaces seem to be related to *reflective Lorentzian lattices* (cf. Dolgachev 2008).
- *Automorphisms* of a $K3$ surfaces induce automorphisms of the intersection form $H^2(K3, \mathbb{Z}) \cong II_{19,3}$ and the Picard sublattice. The Picard lattice can be related equivariantly to the root lattice $II_{25,1}$ of the fake monster Lie algebra. In particular, Mukai's classification (1988) of the symplectic automorphism groups of $K3$-surfaces in terms of certain maximal subgroups of the Mathieu group M_{23} can be understood in this way (Kondō 1998).
- The *equivariant analytical torsion* for a $K3$ surface with involution defines an automorphic form on the moduli space of such $K3$ surfaces which arises as a Borcherds lift of a vector valued modular function related to the fixed point lattice of the involution and twisted denominator identities of the fake monster Lie algebra (Yoshikawa 2004 & 2006). This may be best understood in terms of an equivariant arithmetic Riemann-Roch Theorem (Maillot/Rössler 2008).

Problem: Is there a natural connection between $K3$ surfaces (or other Calabi-Yau spaces) and their automorphisms (or the group of autoequivalences of the bounded derived category of coherent sheaves) to the fake monster Lie algebra or other Lie algebras of this talk?

Possible application: Other Mathematical Physics

Problem: Are the Lie algebras discussed in this talk related to Lie algebras of BPS-states studied in mathematical physics?