McKay's E_7 observation on the Baby Monster

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Abstract

In this paper, we study McKay's E_7 observation on the Baby Monster. By investigating so called derived $c = 7/10$ Virasoro vectors, we show that there is a natural correspondence between dihedral subgroups of the Baby Monster and certain subalgebras of the Baby Monster vertex operator algebra which are constructed by the nodes of the affine E_7 diagram. This allows us to reinterpret McKay's E_7 observation via the theory of vertex operator algebras.

For a class of vertex operator algebras including the Moonshine module, we will show that the product of two Miyamoto involutions associated to derived $c = 7/10$ Virasoro vectors in certain commutant vertex operator algebras is an element of order at most 4. For the case of the Moonshine module, we obtain the Baby monster vertex operator algebra as the commutant and we can identify the group generated by these Miyamoto involutions with the Baby Monster and recover the $\{3, 4\}$-transposition property of the Baby Monster in terms of vertex operator algebras.

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Contents

1 Introduction

The purpose of this article is to give a vertex operator algebra (VOA) theoretical interpretation of McKay's E_7 observation on the Baby Monster. The main idea is to relate certain substructures of the Moonshine VOA $V^\natural$, whose full automorphism group is the Monster $\mathbb{M}$ [B, FLM], to some coset (or commutant) subalgebras related to the Baby Monster. In [LYY1, LYY2, LM], McKay's E_8 observation on the Monster has been studied in detail using the correspondence between 2A involutions of the Monster and simple $c = 1/2$ Virasoro vectors in the Moonshine VOA $V^\natural$ [C, Mi2]. It is established that there exists a natural correspondence between the dihedral groups generated by two 2A-involutions of the Monster and certain sub-VOAs of $V^\natural$ which are constructed naturally by the nodes of the affine E_8 diagram. It turns out that under some general hypotheses [S], the dihedral subalgebras (cf. Section 1.1) associated to the affine E_8 diagram exhaust all possible cases. Therefore, the nine algebras associated to the E_8 diagram are exactly the nine possible subalgebras of $V^\natural$ generated by two simple $c = 1/2$ Virasoro vectors.

In this article, we will study the E_7 observation. We first observe that the Baby Monster acts naturally on a certain commutant subalgebra $V\mathbb{B}^\natural$ of the Moonshine VOA $V^\natural$ which is called the Baby Monster VOA (cf. (5.1)). This observation leads us to study the commutant subalgebra of a simple $c = 1/2$ Virasoro vector in some 2A-subalgebras and certain $c = 7/10$ Virasoro vectors, which we call *derived* Virasoro vectors (cf. Definition 3.5), in $V\mathbb{B}^\natural$. We show that there exists a one-to-one correspondence between 2A involutions of the Baby Monster and derived $c = 7/10$ Virasoro vectors in $V\mathbb{B}^\natural$ (see Theorems 5.9 and 5.13). The main result is a connection between the dihedral groups of the Baby Monster and certain sub-VOAs constructed by the nodes of the affine E_7 diagram in Theorem 5.18. In addition, we show that one can canonically associate involutive automorphisms to derived $c = 7/10$ Virasoro vectors (see Lemma 2.5), which we call σ-involutions, and they satisfy the $\{3, 4\}$-transposition property under certain hypotheses satisfied by a class VOAs containing the Baby Monster VOA $V\mathbb{B}^\natural$ (see Proposition 3.11 in Section 3.2). In order to study the dihedral groups generated by two 2A-elements in the Baby Monster, one needs to study the VOAs generated by two 2A-subalgebras (cf. Section 3) with some conditions, which corresponds to VOAs generated by three simple $c = 1/2$ Virasoro vectors, while in the E_8 case, one only needs to study VOAs generated two simple $c = 1/2$ Virasoro vectors.

In a further paper, we will discuss McKay's E_6 observation on the largest Fischer group [HLY]. Although the general approach to this case is similar, other vertex operator algebras have to be studied and many technical details are different.

To explain our results more precisely, let us review the background of our method and the results established in [LYY1, LYY2, LM]. The main idea is to associate involutions to certain Virasoro vectors of small central charge in $V^\natural$ and $V\mathbb{B}^\natural$.

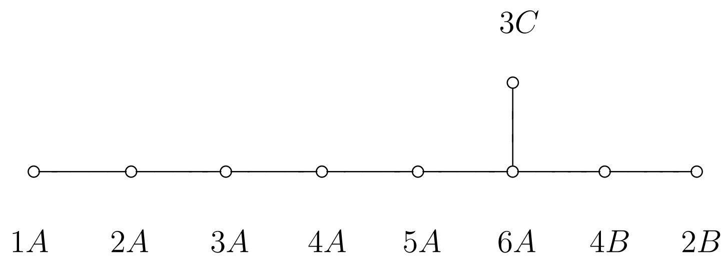
Let R be a simple root lattice with a simply laced root system $\Phi(R)$. We scale R such that the roots have squared length 2. We will consider the lattice VOA $V_{\sqrt{2}R}$ associated to $\sqrt{2}R$. Here and further we will use the standard notation for lattice VOAs as in [FLM]. In [DLMN] Dong et al. constructed a simple Virasoro vector of $V_{\sqrt{2}R}$ of the form

$$\tilde{\omega}_R := \frac{1}{2h(h+2)} \sum_{\alpha\in\Phi(R)} \alpha(-1)^2 \mathbb{1} + \frac{1}{h+2} \sum_{\alpha\in\Phi(R)} e^{\sqrt{2}\alpha}, \tag{1.1}$$

where h denotes the Coxeter number of R. The central charge of $\tilde{\omega}_R$ is 1/2, 7/10 and 6/7 if $R = E_8$, E_7 and E_6, respectively.

1.1 The affine E_8 diagram and the Monster

We will describe some automorphisms of $V_{\sqrt{2}E_8}$ from McKay's E_8 diagram [C, Mc].



(1.2)

Let L_{nX} be a sublattice of E_8 obtained by removing the node labeled nX. Then the index of L_{nX} in E_8 is n and we have a coset decomposition

$$E_8 = \bigsqcup_{j=0}^{n-1} (L_{nX} + j\alpha).$$

Correspondingly, we have a decomposition

$$V_{\sqrt{2}E_8} = \bigoplus_{j=0}^{n-1} V_{\sqrt{2}(L_{nX}+j\alpha)}.$$

Define a linear map ρ_{nX} acting on the component $V_{\sqrt{2}(L_{nX}+j\alpha)}$ by $e^{2\pi\sqrt{-1}\,j/n}$. Then ρ_{nX} is an automorphism of $V_{\sqrt{2}E_8}$ of order n. Now take the simple $c = 1/2$ Virasoro vector $e = \tilde{\omega}_{E_8}$ of $V_{\sqrt{2}E_8}$ defined by (1.1). Denote by U_{nX} the subalgebra of $V_{\sqrt{2}E_8}$ generated by e and $f := \rho_{nX} e$. Note that one can associate involutive automorphisms to these Virasoro vectors via the so-called Miyamoto involutions [Mi1] and therefore the subalgebra U_{nX} represents the symmetry of a dihedral group of order $2n$.

On the other hand, Sakuma [S] showed the following result about subalgebras generated by two simple $c = 1/2$ Virasoro vectors e and f: Let $V = \bigoplus_{n\geq 0} V_n$ be a VOA over $\mathbb{R}$ with $V_0 = \mathbb{R}\mathbb{1}$ and $V_1 = 0$, and assume that the invariant bilinear form on V is positive definite. Then there are exactly nine possible structures for the Griess subalgebra generated by e and f in the degree two subspace V_2 (cf. Theorem 3.8) and they agree with those of the dihedral subalgebras U_{nX} discussed in the previous paragraph. In other words, the dihedral subalgebras U_{nX} exhaust all the possibilities.

In [LYY1, LYY2, LM], the algebra U_{nX} is studied in detail and it is shown that there exists a natural embedding of $U_{nX} \hookrightarrow V^\natural$ for each node nX. This together with Sakuma's theorem explains how the E_8 structure is imposed in the Moonshine VOA. In addition, it is shown in [LYY2] that the product $\tau_e \tau_f$ of the corresponding Miyamoto involutions on $V^\natural$ is exactly in the Monster conjugacy class nX. In fact, the subalgebra U_{nX} exactly corresponds to a dihedral subgroup generated by two 2A-involutions of the Monster via the one-to-one correspondence between the simple $c = 1/2$ Virasoro vectors and 2A-involutions. Thus the E_8-structure on the Moonshine VOA corresponds naturally to the E_8-structure of the Monster as observed by McKay.

1.2 The affine E_7 diagram and the Baby Monster

In this article, we will give a similar correspondence which associates the affine E_7 diagram to the Baby Monster.

The Baby Monster VOA $V\mathbb{B}^\natural$. Let e be a simple $c = 1/2$ Virasoro vector of the Moonshine VOA. Denote by $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e))$ the commutant subalgebra of $\mathrm{Vir}(e)$ in $V^\natural$. It follows from the one-to-one correspondence between simple $c = 1/2$ Virasoro vectors in $V^\natural$ and 2A-involutions of the Monster (cf. Theorem 5.1) that all simple $c = 1/2$ Virasoro vectors of $V^\natural$ are mutually conjugate under the Monster and thus the VOA structure on $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e))$ is independent of $e \in V^\natural$. Denote by τ_e the involution corresponding to a simple $c = 1/2$ Virasoro vector e. The centralizer $C_{\mathbb{M}}(\tau_e)$ is a double cover $2.\mathbb{B}$ of the Baby Monster simple group $\mathbb{B}$ which naturally acts on $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e))$ so that we denote it by $V\mathbb{B}^\natural$ and call it the *Baby Monster VOA*. The Baby Monster VOA as well as its extension to a superalgebra called the shorter Moonshine module was first constructed by one of the authors (G.H.) in [Hö1][1]. It is proved in [Hö2, Y] that the Baby Monster is indeed the full automorphism group of the Baby Monster VOA and therefore the Baby Monster VOA $V\mathbb{B}^\natural$ is probably the most natural object to be considered in the study of the Baby Monster, the second largest of the 26 sporadic groups.

As $\mathbb{B}$ is involved as a double cover $2.\mathbb{B} \subset \mathbb{M}$ in the Monster, we have also an embedding

$$L(1/2, 0) \otimes V\mathbb{B}^\natural \simeq \mathrm{Vir}(e) \otimes \mathrm{Com}_{V^\natural}(\mathrm{Vir}(e)) \hookrightarrow V^\natural, \tag{1.3}$$

where $L(c, 0)$ denotes a simple Virasoro VOA with central charge c. We will show in Proposition 5.4 that every 2A-involution $s \in \mathrm{Aut}(\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e)))$ of the Baby Monster is covered by a 2A-involution $t \in \mathrm{Aut}(V^\natural)$ of the Monster in the sense that $s = t|_{\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e))}$ for some $t \in C_{\mathbb{M}}(\tau_e)$.

We will also show that every 2A-involution of the Baby Monster is induced by a simple $c = 7/10$ Virasoro vector via Miyamoto involutions in Theorem 5.9 and this correspondence actually is one-to-one if we restrict only to simple $c = 7/10$ Virasoro vectors of

[1] In [Hö1] the shorter Moonshine module is denoted by $VB^\natural$. Our $V\mathbb{B}^\natural$ is the even part of the shorter Moonshine module and corresponds to $VB^\natural_{(0)}$ in loc. cit.

σ-*type* in $V\mathbb{B}^\natural$ as shown in Theorem 5.13. (See Definition 2.4 for the definition of simple $c = 7/10$ Virasoro vectors of σ-type.)

McKay's observation. By using the embedding of the E_7 lattice into the E_8 lattice and similar ideas as in [LYY1, LYY2], we will construct a certain sub-VOA $U_{B(nX)}$ of the lattice VOA $V_{\sqrt{2}E_7}$ associated to each node nX of the affine E_7 diagram (cf. Section 4).

We will show that $U_{B(nX)}$ is contained in the VOA $V\mathbb{B}^\natural$ purely by their VOA structures in Theorem 5.18. These sub-VOAs contain pairs of Virasoro vectors, whose central charge are $7/10$. Then, by identifying the Baby Monster with $\mathrm{Aut}(V\mathbb{B}^\natural)$, we also show in Theorem 5.18 that the products of the corresponding σ-involutions belong to the desired conjugacy classes nX in $\mathbb{B}$ using the Atlas [ATLAS]. Thus our embeddings of $U_{B(nX)}$ into $V\mathbb{B}^\natural$, in some sense, encode the E_7 structure into the Baby Monster VOA $V\mathbb{B}^\natural$ and the Baby Monster which are compatible with the original McKay observation.

***N*-transposition property.** A central point for understanding McKay's observations is to describe a product of two involutions. In the Monster, the product of two 2A-involutions has order less than or equal to 6. This fact is known as the 6-*transposition property* of the Monster. This fact can be deduced directly from the character table of the Monster. On the other hand, Sakuma [S] showed that the 6-transposition property can also be viewed as a consequence of symmetries of a vertex operator algebras. In this paper, we will use Sakuma's theorem and deduce that Miyamoto involutions associated to derived $c = 7/10$ Virasoro vectors of the commutant subalgebra satisfy the $\{3, 4\}$-transposition property under the same assumption as in Sakuma's theorem (cf. Propositions 3.11). Applying this result, we can recover the $\{3, 4\}$-transposition property of the Baby Monster. It is true that one still needs to use the character tables to identify the conjugacy classes of the Baby Monster, but it is worth to emphasize that the bound for the order of products of two involutions is given by the theory of vertex operator algebras.

1.3 The organization of the paper

The organization of this article is as follows: In Section 2, we review basic properties about Virasoro VOAs and Virasoro vectors. In Section 3, we study a special vertex operator algebra, which we call the 2A-algebra for the Monster.

In Section 4, we recall the definition of a commutant sub-VOA and define certain commutant subalgebras associated to the root lattice of type E_7 using the method described in [LYY1, LYY2]. It turns out that these commutant subalgebras contain many Virasoro vectors, which will be used to define some involutions in the last section.

In Section 5, the commutant subalgebra $V\mathbb{B}^\natural$ of $V^\natural$ is studied. The full automorphism group of $V\mathbb{B}^\natural$ is shown to be the Baby Monster simple group in [Hö2, Y]. We show that there is a one-to-one correspondence between 2A-involutions of the Baby Monster and simple $c = 7/10$ Virasoro vectors of σ-type in $V\mathbb{B}^\natural$.

Finally, we discuss the embeddings of the commutant subalgebras constructed in Section 4 into $V\mathbb{B}^\natural$ in Section 5.3. We show that the simple $c = 7/10$ Virasoro vectors defined in Section 4.2 can be embedded into $V\mathbb{B}^\natural$. Moreover, the product of the corresponding σ-involutions belongs to the conjugacy class associated to the node.

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Notation and Terminology. In this article, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of non-negative integers, integers, reals and complex numbers, respectively. We denote the ring $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}_p$ with a positive integer p and often identify the integers $0, 1, \dots, p-1$ with their images in $\mathbb{Z}_p$. We denote the Monster simple group by $\mathbb{M}$, the Baby Monster simple group by $\mathbb{B}$.

Every vertex operator algebra is defined over the field $\mathbb{C}$ of complex numbers unless otherwise stated. A VOA V is called *of CFT-type* if it is non-negatively graded $V = \bigoplus_{n\geq 0} V_n$ with $V_0 = \mathbb{C}\mathbb{1}$. For a VOA structure $(V, Y(\cdot, z), \mathbb{1}, \omega)$ on V, the vector ω is called the *conformal vector* of V. For simplicity, we often use (V, ω) to denote the structure $(V, Y(\cdot, z), \mathbb{1}, \omega)$. The vertex operator $Y(a, z)$ of $a \in V$ is expanded such that $Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$.

For c, $h \in \mathbb{C}$, let $L(c, h)$ be the irreducible highest weight module over the Virasoro algebra with central charge c and highest weight h. It is well-known that $L(c, 0)$ has a simple VOA structure [FZ]. An element $e \in V$ is referred to as a *Virasoro vector of central charge* $c_e \in \mathbb{C}$ if $e \in V_2$ and satisfies $e_{(1)}e = 2e$ and $e_{(3)}e = (c_e/2)\cdot\mathbb{1}$. It is well-known that the associated modes $L^e(n) := e_{(n+1)}$, $n \in \mathbb{Z}$, generate a representation of the Virasoro algebra on V (cf. [Mi1]), i.e., they satisfy the commutator relation

$$[L^e(m), L^e(n)] = (m-n)L^e(m+n) + \delta_{m+n,0}\frac{m^3-m}{12}c_e.$$

Therefore, a Virasoro vector together with the vacuum vector generates a Virasoro VOA inside V. We will denote this subalgebra by $\mathrm{Vir}(e)$.

In this paper, we define a sub-VOA of V to be a pair (U, e) of a subalgebra U containing the vacuum element $\mathbb{1}$ and a conformal vector e for U such that (U, e) inherits the grading of V, that is, $U = \bigoplus_{n\geq 0} U_n$ with $U_n = V_n \cap U$, but e may not be the conformal vector of V. In the case that e is also the conformal vector of V, we will call the sub-VOA (U, e) a *full* sub-VOA.

For a positive definite even lattice L, we will denote the lattice VOA associated to L by V_L (cf. [FLM]). We adopt the standard notation for V_L as in [FLM]. In particular, V_L^+ denotes the fixed point subalgebra of V_L under a lift of the (-1)-isometry on L. The letter Λ always denotes the Leech lattice, the unique even unimodular lattice of rank 24 without roots.

Given a group G of automorphisms of V, we denote by V^G the fixed point subalgebra of G in V. The subalgebra V^G is called the *G-orbifold* of V in the literature. For a V-module $(M, Y_M(\cdot, z))$ and $\sigma \in \mathrm{Aut}(V)$, we set ${}^{\sigma}Y_M(a, z) := Y_M(\sigma^{-1}a, z)$ for $a \in V$. Then the *σ-conjugate module* $\sigma \circ M$ of M is defined to be the module structure $(M, {}^{\sigma}Y_M(\cdot, z))$.

2 Virasoro vertex operator algebras

Let

$$\begin{aligned} c_m &:= 1 - \frac{6}{(m+2)(m+3)}, \qquad m = 1, 2, \ldots, \\ h_{r,s}^{(m)} &:= \frac{\{r(m+3) - s(m+2)\}^2 - 1}{4(m+2)(m+3)}, \quad 1 \leq s \leq r \leq m+1. \end{aligned} \tag{2.1}$$

It is shown in [W] that $L(c_m, 0)$ is rational and $L(c_m, h_{r,s}^{(m)})$, $1 \leq s \leq r \leq m+1$, are all irreducible $L(c_m, 0)$-modules (see also [DMZ]). This is the so-called unitary series of the Virasoro VOAs. The fusion rules among $L(c_m, 0)$-modules are computed in [W] and given by

$$L(c_m, h_{r_1,s_1}^{(m)}) \times L(c_m, h_{r_2,s_2}^{(m)}) = \sum_{\substack{i\in I\\ j\in J}} L(c_m, h_{|r_1-r_2|+2i-1,|s_1-s_2|+2j-1}^{(m)}), \tag{2.2}$$

where

$$I = \{1\,, 2,\, \ldots,\, \min\{r_1,\, r_2,\, m+2-r_1,\, m+2-r_2\}\},$$

$$J = \{1,\, 2,\, \ldots,\, \min\{s_1,\, s_2,\, m+3-s_1,\, m+3-s_2\}\}.$$

Definition 2.1. A Virasoro vector x with central charge c is called *simple* if $\mathrm{Vir}(x) \simeq L(c, 0)$. A simple $c = 1/2$ Virasoro vector is called an *Ising vector.*

The fusion rules among $L(c_m, 0)$-modules have a canonical $\mathbb{Z}_2$-symmetry and this symmetry gives rise to an involutive automorphism of a VOA.

Theorem 2.2 ([Mi1]). *Let V be a VOA and $e \in V$ a simple Virasoro vector with central charge c_m. Denote by $V_e[h_{r,s}^{(m)}]$ the sum of irreducible* $\mathrm{Vir}(e) = L(c_m, 0)$*-submodules isomorphic to $L(c_m, h_{r,s}^{(m)})$, $1 \le s \le r \le m+1$. Then the linear map*

$$\tau_e := \begin{cases} (-1)^{r+1} & \text{on } V_e[h_{r,s}^{(m)}] \quad \text{if } m \text{ is even}, \\ (-1)^{s+1} & \text{on } V_e[h_{r,s}^{(m)}] \quad \text{if } m \text{ is odd}, \end{cases}$$

defines an automorphism of V called τ-involution associated to e.

In this paper, we frequently consider simple Virasoro vectors with central charges $c_1 = 1/2$ and $c_2 = 7/10$. Here we recall the definitions of σ-type $c = 1/2$ and $c = 7/10$ Virasoro vectors. The corresponding σ-involutions will also be defined.

Definition 2.3. An Ising vector e of a VOA V is said to be *of σ-type on V* if $\tau_e = \mathrm{id}$ on V, i.e., $V_e[1/16] = 0$.

In this case, one has $V = V_e[0] \oplus V_e[1/2]$ and the map σ_e defined by

$$\sigma_e := \begin{cases} 1 & \text{on } V_e[0], \\ -1 & \text{on } V_e[1/2]. \end{cases}$$

is an automorphism of V (cf. [Mi1]).

Definition 2.4. A simple $c = 7/10$ Virasoro vector u of a VOA V is said to be *of σ-type on V* if $V_u[7/16] = V_u[3/80] = 0$.

Let $u \in V$ be a simple $c = 7/10$ Virasoro vector of σ-type. Then one has the isotypical decomposition

$$V = V_u[0] \oplus V_u[3/2] \oplus V_u[1/10] \oplus V_u[3/5].$$

Define a linear automorphism $\sigma_u \in \mathrm{End}(V)$ by

$$\sigma_u := \begin{cases} 1 & \text{on } V_u[0] \oplus V_u[3/5], \\ -1 & \text{on } V_u[3/2] \oplus V_u[1/10]. \end{cases} \tag{2.3}$$

The fusion rules (2.2) (cf. Theorem 2.2) imply:

Lemma 2.5. *The linear map σ_u is an automorphism of V.*

We also need the following result:

Lemma 2.6. *Let V be a VOA with grading $V = \bigoplus_{n\ge 0} V_n$, $V_0 = \mathbb{C}\mathbb{1}$ and $V_1 = 0$, and $u \in V$ a Virasoro vector such that $\mathrm{Vir}(u) \simeq L(c_m, 0)$. Then the zero mode $o(u) = u_{(1)}$ acts on the Griess algebra of V semisimply with possible eigenvalues 2 and $h^{(m)}_{r,s}$, $1 \le s \le r \le m+1$. Moreover, if $h^{(m)}_{r,s} \ne 2$ for $1 \le s \le r \le m+1$ then the eigenspace for the eigenvalue 2 is one-dimensional, namely, it is spanned by the Virasoro vector u.*

Proof: Since V is a module over a rational VOA $\mathrm{Vir}(u) \simeq L(c_m, 0)$, the zero mode $o(u)$ acts on V semisimply. In the following we use the convention as in Theorem 2.2. Let $v \in V_2$ be an eigenvector with eigenvalue λ. By the linearity, we may assume that $v \in V_u[h^{(m)}_{r,s}]$ with $1 \le s \le r \le m+1$ and $\lambda = h^{(m)}_{r,s} + n$, $n \in \mathbb{N}$. Suppose $n > 0$. If $h^{(m)}_{r,s} \ne 0$, that is, $(r,s) \ne (1,1)$, then $\mathrm{Vir}(u)\cdot v$ contains a non-zero vector of $u_{(1)}$-weight $\lambda - 1$ which belongs to the weight one subspace of V, a contradiction. If $h^{(m)}_{r,s} = 0$, then $\mathrm{Vir}(u)v \cong L(c_m, 0)$ as a $\mathrm{Vir}(u)$-module. That forces $\lambda \le 2$ and hence $\lambda = 0$ or 2. If $n = 2$, then there exists $f \in \mathrm{Hom}_{\mathrm{Vir}(u)}(\mathrm{Vir}(u), V)$ such that $v = f(u)$. In this case $u_{(3)}v = u_{(3)}f(u) = f(u_{(3)}u) = (c_m/2)\cdot f(\mathbb{1})$ is a non-zero vector of the weight zero subspace of V and hence $u_{(3)}v$ is a multiple of the vacuum vector of V. Write $f(\mathbb{1}) = k\mathbb{1}$. Then

$$v = f(u) = f(u_{(-1)}\mathbb{1}) = u_{(-1)}f(\mathbb{1}) = u_{(-1)}\cdot k\mathbb{1} = ku.$$

Therefore, the eigenspace for the eigenvalue 2 is one-dimensional and spanned by u. ∎

Among $L(c_m, 0)$-modules, only $L(c_m, 0)$ and $L(c_m, h^{(m)}_{m+1,1})$ are simple currents, and it is shown in [LLY] that $L(c_m, 0) \oplus L(c_m, h^{(m)}_{m+1,1})$ forms a simple current extension of $L(c_m, 0)$. Note that $h^{(m)}_{m+1,1} = m(m+1)/4$ is an integer if $m \equiv 0,\ 3 \pmod 4$ and a half-integer if $m \equiv 1,\ 2 \pmod 4$.

Theorem 2.7 ([LLY])**.** *(1) The $\mathbb{Z}_2$-graded simple current extension*

$$\mathcal{W}(c_m) := L(c_m, 0) \oplus L(c_m, h^{(m)}_{m+1,1})$$

has a unique simple rational vertex operator algebra structure if $m \equiv 0,\ 3 \pmod 4$, and a unique simple rational vertex operator superalgebra structure if $m \equiv 1,\ 2 \pmod 4$, which extends $L(c_m, 0)$.
(2) Let M be an irreducible $L(c_m, 0)$-module and $\tilde{M} = L(c_m, h^{(m)}_{m+1,1}) \times M$ be the fusion product. If $\tilde{M}$ is not isomorphic to M, then M is uniquely extended to an irreducible either untwisted or $\mathbb{Z}_2$-twisted $\mathcal{W}(c_m)$-module which is given by $M \oplus \tilde{M}$ as an $L(c_m, 0)$-module. If $\tilde{M}$ and M are isomorphic $L(c_m, 0)$-modules, then M affords a structure of an irreducible either untwisted or $\mathbb{Z}_2$-twisted $\mathcal{W}(c_m)$-module on which there are two inequivalent structures. These structures are $\mathbb{Z}_2$-conjugates of each other and we denote them by $M^{\pm}$.

3 The 2A algebra for the Monster

In this section, we will review and list some properties of a certain VOA called the 2A-algebra for the Monster which is related to a dihedral subgroup of the Monster. As an application of this algebra, we will show that certain commutant algebras of the Virasoro VOA $L(1/2,0)$ have a subgroup of automorphisms satisfying the $\{3,4\}$-transposition property. This result will be used in the last section to study the Moonshine VOA and its subalgebra related to the Baby Monster.

By Theorem 2.7, $\mathcal{W}(1/2) = L(1/2,0) \oplus L(1/2,1/2)$ and $\mathcal{W}(7/10) = L(7/10,0) \oplus L(7/10,3/2)$ form simple vertex operator superalgebras. As the even part of a tensor product of these SVOAs

$$U_{2A} := L(1/2,0) \otimes L(7/10,0) \oplus L(1/2,1/2) \otimes L(7/10,3/2).$$

forms a simple VOA. We call U_{2A} the *2A-algebra* for the Monster.

The 2A-algebra can be also constructed along with the recipe described in Section 4 via the embedding $A_1 \oplus E_7 \hookrightarrow E_8$.

The structure as well as the representation theory of U_{2A} is well-studied in [Mi1, LY1]. We first review their results.

Commutant subalgebras. Let V be a VOA and (U,e) a sub-VOA. Then the commutant subalgebra of U is defined by

$$\mathrm{Com}_V(U) := \{a \in V \mid a_{(n)}U = 0 \text{ for all } n \geq 0\}. \tag{3.1}$$

It is known (cf. [FZ, Theorem 5,2]) that

$$\mathrm{Com}_V(U) = \ker_V e_{(0)} \tag{3.2}$$

and in particular $\mathrm{Com}_V(U) = \mathrm{Com}_V(\mathrm{Vir}(e))$. Namely, the commutant subalgebra of U is determined only by the conformal vector e of U. If $\omega_{(2)}e = 0$, it is also shown in Theorem 5.1 of [FZ] that $\omega - e$ is also a Virasoro vector. In that case we have two mutually commuting subalgebras $\mathrm{Com}_V(\mathrm{Vir}(e)) = \ker_V e_{(0)}$ and $\mathrm{Com}_V(\mathrm{Vir}(\omega - e)) = \ker_V(\omega - e)_{(0)}$ and the tensor product $\mathrm{Com}_V(\mathrm{Vir}(\omega-e)) \otimes \mathrm{Com}_V(\mathrm{Vir}(e))$ forms an extension of $\mathrm{Vir}(e) \otimes \mathrm{Vir}(\omega - e)$. If $V_1 = 0$, we always have $\omega_{(2)}e = 0$. More generally, we say a sum $\omega = e^1 + \cdots + e^n$ is a *Virasoro frame* if all e^i are Virasoro vectors and $[Y(e^i,z_1), Y(e^j,z_2)] = 0$ for $i \neq j$.

3.1 Griess algebra and Representation theory

Let ω^1 and ω^2 be the Virasoro vectors of $L(1/2,0) \otimes L(7/10,0) \subset U_{2A}$ with central charges 1/2 and 7/10, respectively, and let $x \in L(1/2,1/2) \otimes L(7/10,3/2) \subset U_{2A}$ be a highest weight vector. It is well-known that the SVOA $\mathcal{W}(1/2) = L(1/2, 0) \oplus L(1/2, 1/2)$ has a realization by a single free fermion [FFR, FRW] and the extension $\mathcal{W}(7/10) = L(7/10, 0) \oplus L(7/10, 3/2)$ is isomorphic to the Neveu-Schwarz SVOA of central charge 7/10 [GKO]. Hence, there exists a presentation

$$\omega^1 = \frac{1}{2}\psi_{-3/2}\psi_{-1/2}\mathbb{1}, \quad \omega^2 = \frac{1}{2}G(-1/2)G(-3/2)\mathbb{1}, \quad x = \psi_{-1/2}\mathbb{1} \otimes G(-3/2)\mathbb{1}$$

with $Y(x, z) = \psi(z) \otimes G(z)$, $\psi(z) = \sum_{n\in\mathbb{Z}} \psi_{n+1/2} z^{-n-1}$, $G(z) = \sum_{r\in\mathbb{Z}+1/2} G(r) z^{-r-3/2}$ such that

$$[\psi_r, \psi_s]_+ = \delta_{r+s,0}, \qquad [\omega^2_{(m+1)}, G(r)] = \left(\frac{1}{2}m - r\right) G(m+r),$$

$$[G(r), G(s)]_+ = 2\omega^2_{(r+s+1)} + \delta_{r+s,0}\frac{7}{30}\left(r^2 - \frac{1}{4}\right). \tag{3.3}$$

By the presentation above, we can calculate the structure of the Griess algebra of U_{2A} as follows (cf. [LYY2]):

$a_{(1)}b$	ω^1	ω^2	x
ω^1	$2\omega^1$	0	$\frac{1}{2}x$
ω^2		$2\omega^2$	$\frac{3}{2}x$
x			$\frac{14}{15}\omega^1 + 2\omega^2$

$\langle a, b\rangle$	ω^1	ω^2	x
ω^1	$\frac{1}{4}$	0	0
ω^2		$\frac{7}{20}$	0
x			$\frac{7}{15}$

(3.4)

Note that $\{\omega^1, \omega^2, x\}$ forms an orthogonal basis of the Griess algebra. By a direct computation, we can verify the following.

Lemma 3.1 ([LYY2])**.** *There exist exactly three Ising vectors in U_{2A}, given by*

$$\omega^1, \quad e^+ := \frac{1}{8}\omega^1 + \frac{5}{8}\omega^2 + \frac{\sqrt{15}}{8}x, \quad e^- := \frac{1}{8}\omega^1 + \frac{5}{8}\omega^2 - \frac{\sqrt{15}}{8}x.$$

These Ising vectors are mutually conjugated by the associated σ-involutions, namely,

$$\sigma_{\omega^1} e^{\pm} = e^{\mp}, \quad \sigma_{e^\pm}\omega^1 = e^{\mp}, \quad \sigma_{e^\pm} e^{\mp} = \omega^1.$$

In particular, $\mathrm{Aut}(U_{2A})$ *is isomorphic to* S_3 *which is generated by σ-type involutions.*

By the lemma above, we see that there exist three Virasoro frames inside U_{2A}:

$$\omega = \omega^1 + \omega^2 = e^+ + f^+ = e^- + f^-, \quad f^\pm := \omega - e^\pm,$$

and these frames are mutually conjugate under the σ-involutions.

Since U_{2A} is a $\mathbb{Z}_2$-graded simple current extension of a rational VOA $L(1/2, 0) \otimes L(7/10, 0)$, U_{2A} is also rational. The classification of irreducible U_{2A}-modules is completed in [LY1].

Theorem 3.2 ([LY1]). *The VOA U_{2A} is rational and there are eight isomorphism types of irreducible modules over U_{2A}. Their shapes as $\mathrm{Vir}(\omega^1)\otimes\mathrm{Vir}(\omega^2)\simeq L(1/2,0)\otimes L(7/10,0)$-modules are as follows:*

$$U(0,0)=[0,0]\oplus[1/2,3/2],\qquad U(1/2,0)=[1/2,0]\oplus[0,3/2],\quad U(0,1/10)=[0,1/10]\oplus[1/2,3/5],$$

$$U(0,3/5)=[0,3/5]\oplus[1/2,1/10],\quad U(1/16,7/16)^{\pm}=[1/16,7/16]^{\pm},\quad U(1/16,3/80)^{\pm}=[1/16,3/80]^{\pm},$$

where $[h_1,h_2]$ denotes $L(1/2,h_1)\otimes L(7/10,h_2)$ and M^- is the contragradient (or dual) module of M^+.

By the list of irreducible modules, we see that U_{2A} is a maximal extension of $L(1/2,0)\otimes L(7/10,0)$ as a simple VOA.

We consider the conjugacy of irreducible U_{2A}-modules under the action of $\mathrm{Aut}(U_{2A})$. By comparing top weights, we see that $g\circ U(h_1,h_2)\simeq U(h_1,h_2)$ for any $g\in\mathrm{Aut}(U_{2A})$ if $(h_1,h_2)=(0,0)$ or $(0,3/5)$. The remaining modules can be divided into two groups having the same top weights, 1/2 and 1/10, each consists of three irreducibles. To determine the isomorphism types of the conjugates of the modules of the form $U(1/16,h)^{\pm}$ with $h=7/16$ and 3/80, we first fix the labeling signs.

By Theorem 2.7, there exist two inequivalent $\mathbb{Z}_2$-twisted $\mathcal{W}(1/2)=L(1/2,0)\oplus L(1/2,1/2)$-module structures on $L(1/2,1/16)$, which we denote by $L(1/2,1/16)^{\pm}$. On $L(1/2,1/16)^{\pm}$, the vertex operator associated to a highest weight vector $\psi_{-1/2}\mathbb{1}$ of $L(1/2,1/2)$ acts as a free fermionic field $\phi(z)=\sum_{n\in\mathbb{Z}}\phi_n z^{-n-1/2}$ such that $[\phi_m,\phi_n]_+=\delta_{m+n,0}$. The operator ϕ_0 acts by $\pm 1/\sqrt{2}$ on the top levels of $L(1/2,1/16)^{\pm}$. Similarly, the $L(7/10,0)$-modules $L(7/10,h)$, $h=7/16$, 3/80, can be extended to irreducible $\mathbb{Z}_2$-twisted modules $L(7/10,h)^{\pm}$ over the Neveu-Schwarz SVOA $\mathcal{W}(7/10)=L(7/10,0)\oplus L(7/10,3/2)$, which form the Ramond sectors. Namely, the vertex operator associated to the highest weight vector $G(-3/2)\mathbb{1}\in L(7/10,3/2)$ has an expression $Y(G(-3/2)\mathbb{1},z)=\sum_{r\in\mathbb{Z}}G(r)z^{-r-3/2}$ such that $[G(r),G(s)]_+$ is given as in (3.3). The operator $G(0)$ acts by $\pm\sqrt{h-\frac{1}{24}\cdot\frac{7}{10}}$ on the top levels of $L(7/10,h)^{\pm}$, $h=7/16$, 3/80.

Let us consider the zero mode action $o(x):=x_{(1)}$ of the highest weight vector $x=\psi_{-1/2}\mathbb{1}\otimes G(-3/2)\mathbb{1}\in U_{2A}$ on the top level of $L(1/2,1/16)\otimes L(7/10,h)$, $h=7/16$, 3/80. Since $o(x)=\phi_0\otimes G(0)$ by the representation, it acts by

$$\pm\frac{1}{\sqrt{2}}\cdot\sqrt{h-\frac{1}{24}\cdot\frac{7}{10}}$$

on the top level. We fix the signs such that $o(x)$ acts by $\pm 7/4\sqrt{15}$ on the top levels of $U(1/16,7/16)^{\pm}$. Then we can define the signs of $U(1/16,3/80)^{\pm}$ by the fusion rule

$$U(1/16,7/16)^{\pm}\times U(0,3/5)=U(1/16,3/80)^{\pm}\tag{3.5}$$

(cf. [LY1, LLY]).

We consider the isomorphism classes of conjugates of irreducible modules with top weight 1/2, namely, $g\circ U(1/2,0)$ and $g\circ U(1/16,7/16)^{\pm}$ for $g\in\{\sigma_{\omega^1},\sigma_{e^+},\sigma_{e^-}\}$. By definition, $\sigma_{\omega^1}\circ U(1/16,7/16)^{\pm}=U(1/16,7/16)^{\mp}$ since σ_{ω^1} coincides with the canonical $\mathbb{Z}_2$-symmetry of $U_{2A}=[0,0]\oplus[1/2,3/2]$. Now consider $\sigma_{e^\pm}\circ U(1/16,7/16)^{\varepsilon}$, $\varepsilon=\pm$. The zero mode action $o(x)$ on the top level of $\sigma_{e^\pm}\circ U(1/16,7/16)^{\varepsilon}$ is equivalent to the zero mode of $\sigma_{e^\pm}x$ on the top level of $U(1/16,7/16)^{\varepsilon}$. We compute $\sigma_{e^\pm}x$. Since $\sigma_{e^\pm}\omega^1=e^{\mp}$, one has

$$\begin{aligned}\omega^1=\sigma_{e^\pm}e^{\mp}\quad&=\frac{1}{8}\sigma_{e^\pm}\omega^1+\frac{5}{8}\sigma_{e^\pm}\omega^2\mp\frac{\sqrt{15}}{8}\sigma_{e^\pm}x\\&=\frac{1}{8}e^{\mp}+\frac{5}{8}f^{\mp}\mp\frac{\sqrt{15}}{8}\sigma_{e^\pm}x.\end{aligned}$$

Solving this, we obtain

$$\sigma_{e^\pm}x=\pm\frac{1}{2\sqrt{15}}(-7\omega^1+5\omega^2)+\frac{1}{2}x.$$

Then the zero mode of $\sigma_{e^\pm}x$ on the top level of $U(1/16,7/16)^{\varepsilon}$ acts by the scalar

$$\pm\frac{1}{2\sqrt{15}}\left(-7\cdot\frac{1}{16}+5\cdot\frac{7}{16}\right)+\varepsilon\cdot\frac{1}{2}\cdot\frac{7}{4\sqrt{15}}.$$

In the above, we identify the signs $\varepsilon=\pm$ with ±1. From this we can determine the isomorphism classes of $\sigma_{e^\pm}\circ U(1/16,7/16)^{\varepsilon}$ and $\sigma_{e^\pm}\circ U(1/2,0)$, and the result is summarized in the following table.

M	$U(1/2,0)$	$U(1/16,7/16)^+$	$U(1/16,7/16)^-$
$\sigma_{\omega^1}\circ M$	$U(1/2,0)$	$U(1/16,7/16)^-$	$U(1/16,7/16)^+$
$\sigma_{e^+}\circ M$	$U(1/16,7/16)^-$	$U(1/16,7/16)^+$	$U(1/2,0)$
$\sigma_{e^-}\circ M$	$U(1/16,7/16)^+$	$U(1/2,0)$	$U(1/16,7/16)^-$

(3.6)

Since a conjugation by an automorphism keeps the fusion rules invariant, by the fusion rules (3.5) and $U(1/2,0)\times U(0,3/5)=U(0,1/10)$ (cf. [LY1, LLY]) we also have the following table.

M	$U(0,1/10)$	$U(1/16,3/80)^+$	$U(1/16,3/80)^-$
$\sigma_{\omega^1}\circ M$	$U(0,1/10)$	$U(1/16,3/80)^-$	$U(1/16,3/80)^+$
$\sigma_{e^+}\circ M$	$U(1/16,3/80)^-$	$U(1/16,3/80)^+$	$U(0,1/10)$
$\sigma_{e^-}\circ M$	$U(1/16,3/80)^+$	$U(0,1/10)$	$U(1/16,3/80)^-$

(3.7)

Remark 3.3. By the table above, we can explicitly compute that the zero mode $o(x)$ acts by $\mp 1/4\sqrt{15}$ on the top levels of $U(1/16, 3/80)^\pm$.

For the τ-involutions associated to U_{2A}, one has the following result:

Theorem 3.4. *Let V be a VOA containing U_{2A} as a sub-VOA. Then as automorphisms on V, the τ-involutions associated to Ising vectors of the subalgebra U_{2A} satisfy the relations of a Kleinian 4-group:*

$$\tau_{\omega^1}\tau_{e^\pm} = \tau_{e^\pm}\tau_{\omega^1} = \tau_{e^\mp} \quad \textit{and} \quad \tau_{e^+}\tau_{e^-} = \tau_{e^-}\tau_{e^+} = \tau_{\omega^1}.$$

Proof: Since U_{2A} is rational, V is a direct sum of irreducible U_{2A}-submodules given in Theorem 3.2. Let M be an irreducible U_{2A}-submodule of V. It is clear that each τ-involution keeps an irreducible U_{2A}-submodule invariant, and the action of τ_{ω^1} on M is manifest by its $\mathrm{Vir}(\omega^1) \otimes \mathrm{Vir}(\omega^2)$-module structure. Since $\tau_{e^\pm} = \tau_{\sigma_{\omega^1} e^\mp} = \tau_{\sigma_{e^\mp}\omega^1}$, the action of $\tau_{e^\pm}$ on M is equivalent to that of τ_{ω^1} on $\sigma_{e^\mp} \circ M$. Thus the action of $\tau_{e^\pm}$ on M is determined by the conjugacy relations given in (3.6) and (3.7) and the following table summarizes the result.

M	$U(0,0)$	$U(1/2,0)$	$U(1/16,7/16)^\pm$	$U(0,3/5)$	$U(0,1/10)$	$U(1/16,3/80)^\pm$
τ_{ω^1}	$+1$	$+1$	-1	$+1$	$+1$	-1
τ_{e^+}	$+1$	-1	± 1	$+1$	-1	± 1
τ_{e^-}	$+1$	-1	∓ 1	$+1$	-1	∓ 1

(3.8)

From the table above one can see the relations as claimed. ∎

3.2 $\{3,4\}$-transposition property of σ-involutions

We consider σ-involutions induced by the 2A-algebra. We refer the reader to Section 2 for the definition of simple $c = 7/10$ Virasoro vectors of σ-type and their corresponding σ-involutions. See Definition 2.4 and Eq. (2.3) for the details.

Let V be a VOA and let $e \in V$ be an Ising vector.

Definition 3.5. A simple $c = 7/10$ Virasoro vector $f \in \mathrm{Com}_V(\mathrm{Vir}(e))$ is called a *derived Virasoro vector with respect to* e if there exists a sub-VOA U of V containing e and f such that U is isomorphic to the 2A-algebra U_{2A} and $e + f$ is the conformal vector of U.

Lemma 3.6. *A derived $c = 7/10$ Virasoro vector $f \in \mathrm{Com}_V(\mathrm{Vir}(e))$ with respect to e is of σ-type on the commutant subalgebra $\mathrm{Com}_V(\mathrm{Vir}(e))$.*

Proof: Assume that V contains a subalgebra U isomorphic to the 2A-algebra U_{2A} as in Definition 3.5. For an irreducible U-module M, we define the space of multiplicities by

$$H_M := \mathrm{Hom}_{U_{2A}}(M, V).$$

Then we have the isotypical decomposition

$$V = \bigoplus_{M \in \mathrm{Irr}(U_{2A})} M \otimes H_M.$$

Consider the subalgebra $\mathrm{Vir}(e) \otimes \mathrm{Vir}(f) \otimes \mathrm{Com}_V(U)$ of V. One has $\mathrm{Vir}(e) \simeq L(1/2, 0)$ and $\mathrm{Vir}(f) \simeq L(7/10, 0)$. By Theorem 3.2, we have the following decomposition:

$$V_e[0] = [0,0] \otimes H_{U(0,0)} \;\oplus\; [0, 3/2] \otimes H_{U(1/2,0)} \;\oplus\; [0, 1/10] \otimes H_{U(0,1/10)} \oplus\; [0, 3/5] \otimes H_{U(0,3/5)}$$

where $[h_1, h_2]$ denotes a $\mathrm{Vir}(e) \otimes \mathrm{Vir}(f)$-module isomorphic to $L(1/2, h_1) \otimes L(7/10, h_2)$. Then we obtain

$$\begin{aligned} \mathrm{Com}_V(\mathrm{Vir}(e)) \;\; = \;\; & L(7/10, 0) \otimes H_{U(0,0)} \;\oplus\; L(7/10, 3/2) \otimes H_{U(1/2,0)} \\ & \oplus\; L(7/10, 1/10) \otimes H_{U(0,1/10)} \;\oplus\; L(7/10, 3/5) \otimes H_{U(0,3/5)}. \end{aligned} \tag{3.9}$$

Thus f is of σ-type on $\mathrm{Com}_V(\mathrm{Vir}(e))$. ∎

We consider the one-point stabilizer

$$\mathrm{Stab}_{\mathrm{Aut}(V)}(e) := \{g \in \mathrm{Aut}(V) \mid ge = e\}. \tag{3.10}$$

It is clear that $\mathrm{Stab}_{\mathrm{Aut}(V)}(e)$ forms a subgroup of $\mathrm{Aut}(V)$. Each $g \in \mathrm{Stab}_{\mathrm{Aut}(V)}(e)$ keeps the isotypical component $V_e[h]$, $h \in \{0, 1/2, 1/16\}$, invariant so that by restriction we obtain a group homomorphism

$$\begin{array}{rccc} \varphi_e : & \mathrm{Stab}_{\mathrm{Aut}(V)}(e) & \longrightarrow & \mathrm{Aut}(\mathrm{Com}_V(\mathrm{Vir}(e))), \\ & g & \longmapsto & g|_{\mathrm{Com}_V(\mathrm{Vir}(e))}. \end{array} \tag{3.11}$$

Let $f \in \mathrm{Com}_V(\mathrm{Vir}(e))$ be a derived Virasoro vector with respect e. By Lemma 3.6 and Lemma 2.5 above, we obtain an involution $\sigma_f \in \mathrm{Aut}(\mathrm{Com}_V(\mathrm{Vir}(e)))$. Inside U, we have the three Ising vectors e, e^+ and $e^- = \sigma_e e^+$.

Lemma 3.7. *$\varphi_e(\tau_{e^+}) = \varphi_e(\tau_{e^-}) = \sigma_f$ in $\mathrm{Aut}(\mathrm{Com}_V(\mathrm{Vir}(e)))$.*

Proof: By Theorem 3.4, we see that τ_{e^+} and τ_{e^-} are in $\mathrm{Stab}_{\mathrm{Aut}(V)}(e)$. More precisely, we see from (3.8) that both τ_{e^+} and τ_{e^-} act by ± 1 on each isotypical component $M \otimes H_M$ for $M \in \mathrm{Irr}(U_{2A})$. In particular, by (3.8) and (3.9), we see that $\varphi_e(\tau_{e^+})$, $\varphi_e(\tau_{e^-})$ and σ_f define the same automorphism on the commutant subalgebra $\mathrm{Com}_V(\mathrm{Vir}(e))$. ∎

We say a VOA W over $\mathbb{R}$ is *compact* if W has a positive definite invariant bilinear form. We recall the following interesting theorem of Sakuma.

Theorem 3.8 ([S]). *Let W be a VOA over $\mathbb{R}$ with grading $W = \bigoplus_{n\geq 0} W_n$, $W_0 = \mathbb{R}\mathbb{1}$ and $W_1 = 0$, and assume W is compact. Let x, y be Ising vectors in W and denote by $U(x, y)$ the subalgebra of W generated by x and y. Then:*
(1) The 6-transposition property $|\tau_x\tau_y| \leq 6$ holds on W.
(2) The Griess algebra on the weight two subspace $U(x, y)_2$ of $U(x, y)$ is one of nine inequivalent structures.
(3) The Griess algebra structure on $U(x, y)_2$ is unique if $|\tau_x\tau_y| = 5$, 6.
(4) If $|\tau_x\tau_y| = 2$ then $\tau_x y = y$ and $\tau_y x = x$. Moreover, either $\langle x, y\rangle = 0$ or $\langle x, y\rangle = 1/32$. If $\langle x, y\rangle = 1/32$, then x and y generate a copy of the 2A-algebra and we have $\sigma_x y = \sigma_y x$ inside the fixed point subalgebra $W^{\langle\tau_x,\tau_y\rangle}$.

Let I be the set of all derived $c = 7/10$ Virasoro vectors of $\mathrm{Com}_V(\mathrm{Vir}(e))$ with respect to e. We have seen in Lemma 3.6 that each vector of I is of σ-type on $\mathrm{Com}_V(\mathrm{Vir}(e))$. Based on Sakuma's theorem, we will prove that the set of involutions

$$\{\sigma_u \in \mathrm{Aut}(\mathrm{Com}_V(\mathrm{Vir}(e))) \mid u \in I\}$$

satisfies the so-called $\{3, 4\}$-transposition property.

First we will prove a result about regular tetrahedrons of Ising vectors of σ-type.

Lemma 3.9. *Suppose that V is a VOA with $V_1 = 0$ and V has a compact real form $V_{\mathbb{R}}$, that is, V as an $\mathbb{R}$-algebra has a compact $\mathbb{R}$-subalgebra $V_{\mathbb{R}}$ such that $V = \mathbb{C} \otimes_{\mathbb{R}} V_{\mathbb{R}}$. Let a^1, a^2, a^3 and b be Ising vectors of $V_{\mathbb{R}}$ of σ-type on V such that a^1, a^2, a^3 are the three Ising vectors of a 2A-algebra. Then it is impossible that $\langle a^i, b\rangle = 1/32$ for all $1 \leq i \leq 3$.*

Proof: Suppose otherwise and $\langle a^i, b\rangle = 1/32$ for all $1 \leq i \leq 3$. By Lemma 3.1, the σ-involution σ_{a^1}, σ_{a^2} and σ_{a^3} generate a group $H \simeq \mathrm{S}_3$ in $\mathrm{Aut}(V_{\mathbb{R}})$. Using Theorem 3.8, we know that for each $i = 1, 2, 3$, the vectors a^i and b also generate a 2A-algebra and therefore σ_{a^i} and σ_b generate a subgroup $H_i \simeq \mathrm{S}_3$ in $\mathrm{Aut}(V_{\mathbb{R}})$. In particular, $[\sigma_b, \sigma_{a^i}] \neq 1$ for $i = 1, 2, 3$. Note that $H \cap H_i \neq 1$.

Let K be the subgroup of $\mathrm{Aut}(V)$ generated by $\{\sigma_{a^1}, \sigma_{a^2}, \sigma_{a^3}, \sigma_b\}$. It is shown in [Ma2] that the group K is a 3-transposition group of symplectic type, which means, the group generated by two (non-commuting) subgroups H and H_i, both isomorphic to S_3, is isomorphic to either S_3 if $H = H_i$ or S_4 if $H \neq H_i$.

Assume first that $K = \langle H, H_i\rangle$ is isomorphic to S_4. As elements of subgroups isomorphic to S_3 the σ-involutions σ_{a^1}, σ_{a^2}, σ_{a^3} and σ_b would be transpositions. However, given four transpositions in S_4, there are at least two which commute. This is a contradiction. Therefore K must be S_3, which implies that σ_b coincides with one of the σ_{a^i}, $i = 1, 2, 3$. This contradicts that σ_b and σ_{a^i} generate a S_3 for $i = 1, 2, 3$. Therefore the lemma follows. ∎

Remark 3.10. It is known [ATLAS] that there is no 2A-pure elementary abelian subgroup of order 8 in the Monster. We have essentially shown this fact by the theory of vertex operator algebras in Lemma 3.9.

Proposition 3.11. *Suppose that V is a VOA with $V_1 = 0$ and V has a compact real form $V_{\mathbb{R}}$ such that all the Ising vectors of V are in $V_{\mathbb{R}}$. Then for any u, $v \in I$, we have $|\sigma_u\sigma_v| \le 4$ on $\mathrm{Com}_V(\mathrm{Vir}(e))$.*

Proof: First, we note that all derived $c = 7/10$ Virasoro vectors in I are in the real form $V_{\mathbb{R}}$, as it is a $\mathbb{R}$-linear combination of real Ising vectors by Lemma 3.1. Take u, $v \in I$ and consider involutions σ_u and σ_v defined on $\mathrm{Com}_V(\mathrm{Vir}(e))$. By definition of σ-involution, both σ_u and σ_v preserve the real form $V_{\mathbb{R}}$ and therefore the order of $\sigma_u\sigma_v$ on V is the same as that on $V_{\mathbb{R}}$. By definition of derived vectors, there exist subalgebras U^1, U^2 of V isomorphic to the 2A-subalgebras such that $e + u$ and $e + v$ are Virasoro frames of U^1 and U^2, respectively. By Lemma 3.7 there exist Ising vectors $e^1 \in U^1$ and $e^2 \in U^2$ such that $\varphi_e(\tau_{e^1}) = \sigma_u$ and $\varphi_e(\tau_{e^2}) = \sigma_v$. Since φ_e is a group homomorphism, the order $|\sigma_u\sigma_v| = |\varphi_e(\tau_{e^1}\tau_{e^2})|$ divides $|\tau_{e^1}\tau_{e^2}|$. By Theorem 3.8, we know that $|\tau_{e^1}\tau_{e^2}| \le 6$ and we have nothing to prove if $|\tau_{e^1}\tau_{e^2}| \le 4$. So we assume $|\tau_{e^1}\tau_{e^2}| = 5$ or 6.

Case $|\tau_{e^1}\tau_{e^2}| = 5$: Suppose $|\tau_{e^1}\tau_{e^2}| = 5$. Since e^1 is an Ising vector of U^1 which is isomorphic to the 2A-algebra, we can take by Theorem 3.4 another Ising vector $e^{1\prime} \in U^1$ such that $\tau_{e^{1\prime}} = \tau_e\tau_{e^1}$. Since e and e^2 are in another 2A-subalgebra U^2, we see that τ_e and τ_{e^2} commute, from which we obtain $|\tau_{e^{1\prime}}\tau_{e^2}| = |\tau_e\tau_{e^1}\tau_{e^2}| = 10$. This contradicts Sakuma's theorem. Therefore, $|\tau_{e^1}\tau_{e^2}| \ne 5$ and this case is impossible.

Case $|\tau_{e^1}\tau_{e^2}| = 6$: In this case the Griess algebra structure of the subalgebra generated by e^1 and e^2 is unique by Sakuma's theorem and is isomorphic to the 6A-algebra discussed in Appendix A. By the uniqueness, we can identify e^1 and e^2 with those in Appendix A. Below we will freely use the results there. Set $w := \tau_{e^2}\tau_{e^1}e^2$. Then w is an Ising vector which is denoted by e^4 in U_{6A} by (A.2). Since $\langle e^1, w\rangle = 1/32$ by (A.1), the sub-VOA generated by e^1 and w is isomorphic to the 2A-algebra U_{2A}. The third Ising vector x in this 2A-algebra is given by the equation

$$x := e^1 + w - 4w_{(1)}e^1 = \sigma_{e^1}w \tag{3.12}$$

and is corresponding to ω^1 in U_{6A} by (A.3). Moreover, since $\langle e, e^1\rangle = \langle e, e^2\rangle = 1/32$ and $e \in V^{\langle \tau_{e^1}, \tau_{e^2}\rangle}$, we have

$$\langle e, w\rangle = \langle e, \tau_{e^2}\tau_{e^1}e^2\rangle = \langle \tau_{e^1}\tau_{e^2}e, e^2\rangle = \langle e, e^2\rangle = \frac{1}{32}.$$

Therefore, e and w also generate a 2A-algebra.

For the τ-involutions we obtain:

$$\tau_w = \tau_{\tau_{e^2}\tau_{e^1}e^2} = \tau_{e^2}\tau_{e^1}\tau_{e^2}\tau_{e^1}\tau_{e^2} \tag{3.13}$$

as $\tau_{ge^2} = g\tau_{e^2}g^{-1}$ for $g \in \mathrm{Aut}(V)$. It follows from $(\tau_{e^1}\tau_{e^2})^6 = 1$ that τ_w commutes with τ_{e^1}. It follows from (3.13) and (A.4) that

$$\tau_x = \tau_{e^1}\tau_w = (\tau_{e^1}\tau_{e^2})^3 \quad \text{on} \quad V. \tag{3.14}$$

Thus τ_e commutes with both τ_w and τ_x in $\mathrm{Aut}(V)$ since τ_e commutes with both τ_{e^1} and τ_{e^2}.

Now either $|\tau_e\tau_{e^1}\tau_{e^2}| = 3$ or 6 occurs.

Claim: $|\tau_e\tau_{e^1}\tau_{e^2}| = 3$.

Suppose otherwise and $|\tau_e\tau_{e^1}\tau_{e^2}| = 6$. For the third Ising vector $e^{1'}$ in the 2A-algebra U^1 generated by e and e^1 one has $\tau_{e^{1'}} = \tau_e\tau_{e^1}$ on V. Then $\tau_e\tau_{e^1}\tau_{e^2} = \tau_{e^{1'}}\tau_{e^2}$ is of order 6 and hence $e^{1'}$ and e^2 generate a subalgebra isomorphic to the 6A-algebra. Again we have Ising vectors $w' := \tau_{e^2}\tau_{e^{1'}}e^2 = \tau_{e^2}\tau_{e^1}\tau_e e^2 = \tau_{e^2}\tau_{e^1}e^2 = w$ and $x' = \sigma_{e^{1'}}w$ corresponding to e^4 and ω^1 in U_{6A}, respectively, such that $e^{1'}$, $w' = w$ and x' generate a 2A-algebra.

Note that if we set $G = \langle \tau_e, \tau_{e^1}, \tau_w \rangle$ then e, e^1, w and x are Ising vectors of V^G which are of σ-type on it. As $\langle w', e^{1'} \rangle = 1/32$ and e and w generate a 2A-algebra, the following holds in V^G:

$$\frac{1}{32} = \langle w', e^{1'} \rangle = \langle w, e^{1'} \rangle = \langle w, \sigma_e e^1 \rangle = \langle \sigma_e w, e^1 \rangle = \langle \sigma_w e, e^1 \rangle = \langle e, \sigma_w e^1 \rangle = \langle e, x \rangle. \tag{3.15}$$

Therefore, if $|\tau_e\tau_{e^1}\tau_{e^2}| = 6$, then we obtain four σ-type Ising vectors e, e^1, w and x in V^G such that e^1, w and x generate a 2A-algebra and

$$\langle e, e^1 \rangle = \langle e, w \rangle = \langle e, x \rangle = \frac{1}{32}. \tag{3.16}$$

By Lemma 3.9 above, such a configuration is impossible. Thus, $\tau_e\tau_{e^1}\tau_{e^2}$ cannot have order 6 and the claim follows.

Since $\tau_e \in \ker \varphi_e$, it follows from the claim above that $\sigma_u\sigma_v = \varphi_e(\tau_{e^1}\tau_{e^2}) = \varphi_e(\tau_e\tau_{e^1}\tau_{e^2})$ is of order 3 if $|\tau_{e^1}\tau_{e^2}| = 6$. This completes the proof of Proposition 3.11. ∎

4 Commutant subalgebras associated to the root lattice E_7

In this section, we will construct sub-VOAs of the lattice VOA $V_{\sqrt{2}E_7}$ which will correspond to dihedral subgroups of the Baby Monster. We will use the standard notation for lattice VOAs as in [FLM]. Our construction is similar to the construction in [LYY1] in the case of the root lattice E_8 and works in fact for any root lattice of type A_n, D_n or E_n. We start by describing our construction in general and then specialize to the case of the root lattice of type E_7.

4.1 Definition of the subalgebras

The algebras $U(i)$. Let R be a root lattice of type A_n ($n \geq 1$), D_n ($n \geq 4$) or E_n ($n = 6, 7, 8$) and let $\alpha_1, \ldots, \alpha_n$ be a system of simple roots for R. We let α_0 be the root such that $-\alpha_0 = \sum_{i=1}^n m_i \alpha_i$ is the highest root for the chosen simple roots. Note that all m_i are positive integers. We also set $m_0 = 1$. For any $i = 0, \ldots, n$, we consider the sublattice L_i of R generated by the roots α_j, $0 \leq j \leq n$, $j \neq i$. One observes that L_i is also of rank n and the quotient group R/L_i is cyclic of order m_i with generator $\alpha_i + L_i$. Thus one has

$$R = L_i \sqcup (\alpha_i + L_i) \sqcup (2\alpha_i + L_i) \sqcup \cdots \sqcup ((m_i - 1)\alpha_i + L_i). \tag{4.1}$$

We denote by $R_1, \ldots, R_\ell$ the indecomposable components of the lattice L_i which are again root lattices of type A_n, D_n or E_n. Hence $L_i = R_1 \oplus \cdots \oplus R_\ell$ where the direct sum of lattices denotes the orthogonal sum. In fact, the Dynkin diagram of L_i is obtained from the affine Dynkin diagram of R by removing the node α_i and the adjacent edges. We recall here that the affine Dynkin diagram of R is the graph with vertex set $\{\alpha_0, \ldots, \alpha_n\}$ and two nodes α_i and α_j, $0 \leq i, j \leq n$, are joined by an edge if $\langle \alpha_i, \alpha_j \rangle = -1$.

The decomposition (4.1) of the lattice R leads to the decomposition

$$V_{\sqrt{2}R} = \bigoplus_{r=0}^{m_i - 1} V_{\sqrt{2}(r\alpha_i + L_i)}$$

of the lattice VOA $V_{\sqrt{2}R}$. We define a linear map $\rho_i : V_{\sqrt{2}R} \to V_{\sqrt{2}R}$ by

$$\rho_i(u) = \zeta_{m_i}^r u \quad \text{for } u \in V_{\sqrt{2}(r\alpha_i + L_i)}, \qquad \text{where } \zeta_{m_i} = e^{2\pi\sqrt{-1}/m_i}. \tag{4.2}$$

Then ρ_i is an element of $\mathrm{Aut}(V_{\sqrt{2}R})$ of order m_i and the fixed point sub-VOA $V_{\sqrt{2}R}^{\langle \rho_i \rangle}$ is exactly $V_{\sqrt{2}L_i}$.

For a root lattice S we denote by $\Phi(S)$ its root system. Then by [DLMN] the conformal vector ω_R of $V_{\sqrt{2}R}$ is given by

$$\omega_R = \frac{1}{4h} \sum_{\alpha \in \Phi(R)} \alpha(-1)^2 \mathbb{1},$$

where h is the Coxeter number of R. Now define

$$\tilde{\omega}_R := \frac{2}{h+2} \omega_R + \frac{1}{h+2} \sum_{\alpha \in \Phi(R)} e^{\sqrt{2}\alpha}. \tag{4.3}$$

It is shown in [DLMN] that $\tilde{\omega}_R$ is a Virasoro vector of central charge $2n/(n+3)$ if R is of type A_n, 1 if R is of type D_n and $6/7$, $7/10$ and $1/2$ if R is of type E_6, E_7 and E_8, respectively. From the irreducible decomposition $L_i = R_1 \oplus \cdots \oplus R_\ell \subset R$ we have sublattices R_s of R and obtain a factorization

$$V_{\sqrt{2}L_i} = V_{\sqrt{2}R_1} \otimes \cdots \otimes V_{\sqrt{2}R_\ell} \subset V_{\sqrt{2}R}. \tag{4.4}$$

Associated to the root subsystems $\Phi(R_s)$ of $\Phi(R)$, we also have simple Virasoro vectors

$$\omega^s = \tilde{\omega}_{R_s} = \frac{2}{h_s+2}\,\omega_{R_s} + \frac{1}{h_s+2}\sum_{\alpha\in\Phi(R_s)} e^{\sqrt{2}\alpha} \in V_{\sqrt{2}R_s} \subset V_{\sqrt{2}R}, \qquad 1 \le s \le \ell, \tag{4.5}$$

where ω_{R_s} is the conformal vector of $V_{\sqrt{2}R_s}$ and h_s is the Coxeter number of R_s. It follows from the definition that the ω^s are mutually orthogonal simple Virasoro vectors in $V_{\sqrt{2}R}$. Consider

$$X^r := \sum_{\substack{\beta\in r\alpha_i+L_i\\ \langle\beta,\beta\rangle=2}} e^{\sqrt{2}\beta}, \qquad 1 \le r \le m_i - 1,$$

in the weight two subspace of $V_{\sqrt{2}R}$. It is shown in Proposition 2.2 of [LYY1] that the vectors X^r are highest weight vectors for $\mathrm{Vir}(\omega^1)\otimes\cdots\otimes\mathrm{Vir}(\omega^\ell)$ with total weight 2.

Since all ω^s, $1 \le s \le \ell$, are contained in the fixed point sub-VOA $V^+_{\sqrt{2}R}$, which has a trivial weight one subspace, $\omega_R - (\omega^1 + \cdots + \omega^\ell)$ is a Virasoro vector of $V_{\sqrt{2}R}$ as discussed at the beginning of chapter 3. We are interested in the commutant subalgebras defined by

$$\begin{aligned} U(i) &:= \mathrm{Com}_{V_{\sqrt{2}R}}(\mathrm{Vir}(\omega_R - (\omega^1 + \cdots + \omega^\ell))) \\ &= \ker_{V_{\sqrt{2}R}} \left(\omega_R - (\omega^1 + \cdots + \omega^\ell)\right)_{(0)} \end{aligned} \tag{4.6}$$

in the case of $R = E_7$. (The case $R = E_8$ is considered in [LYY1, LYY2].) It is clear from the construction that $U(i)$ has a Virasoro frame $\omega^1 + \cdots + \omega^\ell$. We will consider an embedding of $U(i)$ into a larger VOA and then describe the commutant algebra $U(i)$ using the larger VOA.

It is clear that $U(i)$ forms an extension of the tensor product $\mathrm{Vir}(\omega^1)\otimes\cdots\otimes\mathrm{Vir}(\omega^\ell)$ and contains X^r, $1 \le r \le m_i - 1$. We will see in Section 5 that we can embed $U(i)$ into the Moonshine VOA and therefore $U(i)$ has a trivial weight one subspace. Consequently, the weight two subspace of $U(i)$ carries a structure of a commutative non-associative algebra called the Griess algebra of $U(i)$, even though $V_{\sqrt{2}R}$ has a non-trivial weight one subspace. Below we will explicitly describe the Griess algebra of $U(i)$. Namely, we will see that the Griess algebra of $U(i)$ is given by

$$\mathcal{G}(i) := \mathrm{span}_{\mathbb{C}}\{\ \omega^s,\ X^r \mid 1 \le s \le \ell,\ \ 1 \le r \le m_i - 1\}$$

which is of dimension $\ell + m_i - 1$.

By definition, it is clear that $\tilde{\omega}_R$ and $\rho_i\tilde{\omega}_R$, where $\rho_i \in \mathrm{Aut}(V_{\sqrt{2}R})$ is defined as in (4.2), are linear combinations of ω^s and X^r and hence are contained in $\mathcal{G}(i) \subset U(i)$. We will discuss the structure of the subalgebra generated by $\tilde{\omega}_R$ and $\rho_i\tilde{\omega}_R$.

The algebras $V(i)$. Let in the following $R = E_7$ and fix an embedding of R into E_8. Let

$$Q(R) := \mathrm{Ann}_{E_8}(R) = \{\alpha \in E_8 \mid \langle \alpha, R\rangle = 0\}. \tag{4.7}$$

Then $Q(E_7) \simeq A_1$ and $Q(R) \oplus R$ forms a full rank sublattice of E_8. Note that such an embedding is unique up to an automorphism of E_8.

Recall that L_i is the sublattice of R generated by roots α_j, $j \neq i$. Then we have an embedding of $\tilde{L}_i := Q(R) \oplus L_i$ into E_8. Since L_i is a full rank sublattice of R, $\tilde{L}_i$ is also a full rank sublattice of E_8. Thus $E_8/\tilde{L}_i$ is a finite abelian group whose order is $2m_i$. We fix the corresponding embedding $V_{\sqrt{2}\tilde{L}_i} \subset V_{\sqrt{2}E_8}$.

We have the decomposition $\tilde{L}_i = Q(R) \oplus R_1 \oplus \cdots \oplus R_\ell$ into a sum of irreducible root lattices, which gives rise to a factorization

$$V_{\sqrt{2}\tilde{L}_i} = V_{\sqrt{2}Q(R)} \otimes V_{\sqrt{2}L_i} = V_{\sqrt{2}Q(R)} \otimes V_{\sqrt{2}R_1} \otimes \cdots \otimes V_{\sqrt{2}R_\ell} \subset V_{\sqrt{2}E_8}.$$

Let ω_{E_8} be the conformal vector of $V_{\sqrt{2}E_8}$ and let $\tilde{\omega}_{Q(R)} \in V_{\sqrt{2}Q(R)}$ and $\omega^s \in V_{\sqrt{2}R_s}$ be the Virasoro vectors defined as in (4.3) and (4.5), respectively. Since $\omega_{E_8} - (\tilde{\omega}_{Q(R)} + \omega^1 + \cdots + \omega^\ell)$ is by the same argument as for $U(i)$ a Virasoro vector of $V_{\sqrt{2}E_8}$, we can define a commutant subalgebra

$$V(i) := \mathrm{Com}_{V_{\sqrt{2}E_8}}(\mathrm{Vir}(\omega_{E_8} - (\tilde{\omega}_{Q(R)} + \omega^1 + \cdots + \omega^\ell))). \tag{4.8}$$

Remark 4.1. We note that $\mathrm{Com}_{V(i)}(\mathrm{Vir}(\tilde{\omega}_{Q(R)}))$ coincides with $U(i)$ by the definition of commutant subalgebras. For, as we have orthogonal decompositions $\omega_{E_8} = \omega_{Q(R)} + \omega_R$, $\omega_{Q(R)} = (\omega_{Q(R)} - \tilde{\omega}_{Q(R)}) + \tilde{\omega}_{Q(R)}$ and $\omega_R = (\omega_R - (\omega^1 + \cdots + \omega^\ell)) + (\omega^1 + \cdots + \omega^\ell)$, we have $\mathrm{Com}_{V(i)}(\mathrm{Vir}(\tilde{\omega}_{Q(R)})) \subset \mathrm{Com}_{V_{\sqrt{2}E_8}}(\mathrm{Vir}(\omega_{Q(R)})) = V_{\sqrt{2}R}$. Since $U(i)$ is the maximal sub-VOA of $V_{\sqrt{2}R}$ having $\omega^1 + \cdots + \omega^\ell$ as its conformal vector (cf. discussion in Section 3), one has $\mathrm{Com}_{V(i)}(\mathrm{Vir}(\tilde{\omega}_{Q(R)})) \subset U(i)$. On the other hand, it is clear from the definition of $V(i)$ that $\mathrm{Vir}(\tilde{\omega}_{Q(R)}) \otimes U(i) \subset V(i)$ and hence $U(i) \subset \mathrm{Com}_{V(i)}(\mathrm{Vir}(\tilde{\omega}_{Q(R)}))$.

We finally set

$$\mathcal{F}(i) := \{g \in \mathrm{Aut}(V_{\sqrt{2}E_8}) \mid g = \mathrm{id} \ \text{ on } \ V_{\sqrt{2}\tilde{L}_i}\}. \tag{4.9}$$

Then $\mathcal{F}(i)$ is canonically isomorphic to the group of characters of $E_8/\tilde{L}_i$. The subalgebra $V(i)$ of $V_{\sqrt{2}E_8}$ is invariant under the action of $\mathcal{F}(i)$ since all $\tilde{\omega}_{Q(R)}$, ω^1, $\ldots$, ω^ℓ and the conformal vector ω_{E_8} of $V_{\sqrt{2}E_8}$ are clearly fixed by $\mathcal{F}(i)$. Note that the special Ising vector

$$\hat{e} := \tilde{\omega}_{E_8} = \frac{1}{16}\omega_{E_8} + \frac{1}{32}\sum_{\alpha\in\Phi(E_8)} e^{\sqrt{2}\alpha} \in V_{\sqrt{2}E_8} \tag{4.10}$$

is contained in $V(i)$ (cf. [LYY1, LYY2]) and thus

$$\{g\hat{e} \mid g \in \mathcal{F}(i)\} \subset V(i).$$

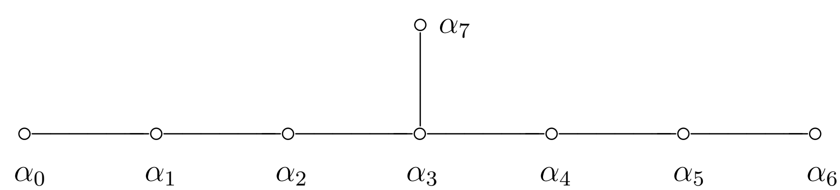
Remark 4.2. Here is a brief explanation of the rôles of the algebras $V(i)$, $U(i)$ and $\mathcal{G}(i)$ in this article. Later we will consider the commutant subalgebra $V\mathbb{B}^\natural$ of the Moonshine VOA $V^\natural$. Then the sub-VOA $V(i)$ corresponds to the one generated by Ising vectors in $V^\natural$ under a certain configuration, and the sub-VOA $U(i)$ of $V(i)$ describes the subspace of $V(i)$ contained in $V\mathbb{B}^\natural$. The structure of the Griess algebras $\mathcal{G}(i)$ would correspond to those in Table 3 of [C] and one can easily relate $\mathcal{G}(i)$ with the E_7-structure as in the case of E_8 [LYY1].

Remark 4.3. Recall that the central charge of the simple Virasoro vector $\tilde{\omega}_R$ is $7/10$. We will show in Section 5 that there is indeed a nice correspondence between 2A-involutions of the Baby Monster and simple $c = 7/10$ Virasoro vectors of σ-type in the subalgebra $V\mathbb{B}^\natural$ of $V^\natural$.

4.2 Explicit description of the subalgebras

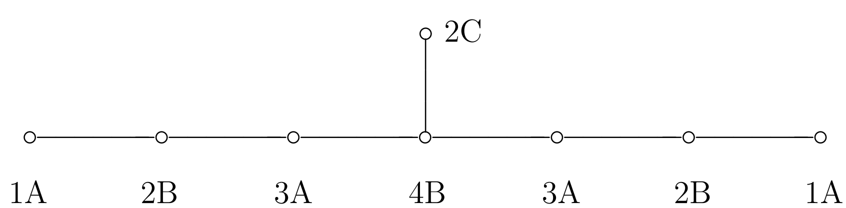
In this subsection we study the case where R is a root lattice of type E_7 in detail.

The affine Dynkin diagram of a root lattice of type E_7 is the following graph:



(4.11)

We like to explain McKay's correspondence [Mc] between the numerical labels m_i of the affine E_7 Dynkin diagram and the Baby Monster conjugacy classes 1A, 2B, 2C, 3A and 4B into which the product of two 2A-involutions of the Baby Monster falls as given by the following figure:



(4.12)

Note that the correspondence is not one-to-one but only up to diagram automorphism.

To specialize to this situation, we change our notation slightly and denote L_i by L_{nX}, ρ_i by ρ_{nX}, $\tilde{L}_i$ by $\tilde{L}_{nX}$, $\mathcal{F}(i)$ by $\mathcal{F}_{nX}$, $V(i)$ by $V_{B(nX)}$, $U(i)$ by $U_{B(nX)}$ and $\mathcal{G}(i)$ by $\mathcal{G}_{B(nX)}$, respectively, where $nX \in \{1A, 2B, 2C, 3A, 4B\}$ is the label of the corresponding node in (4.12). Explicitly, we have

$$L_{1A} \simeq E_7, \quad L_{2B} \simeq A_1 \oplus D_6, \quad L_{2C} \simeq A_7, \quad L_{3A} \simeq A_2 \oplus A_5, \quad L_{4B} \simeq A_1 \oplus A_3 \oplus A_3. \tag{4.13}$$

Remark 4.4. Apparently, there are two distinct nodes (up to the diagram automorphism) that have the same numerical label 2. Therefore, there is some ambiguity for the assignment of the $2B$ and $2C$ labels. However, the sublattice structure $L_{4B} \subset L_{2B} \subset L_{1A}$ justify our labeling. Note that $L_{2C} \simeq A_7$ does not contain a sublattice isometric to L_{4B}. The above inclusions correspond to the power map $(4B)^2 = 2B$ between conjugacy classes of the Baby Monster, and with this rule, the assignment is uniquely determined.

Let $\tilde{\alpha} \in E_8$ be a root so that $Q(E_7) = \mathbb{Z}\tilde{\alpha} \simeq A_1$ and

$$\tilde{\omega}_{Q(E_7)} = \omega^+(\sqrt{2}\tilde{\alpha}) = \frac{1}{8}\tilde{\alpha}(-1)^2\mathbb{1} + \frac{1}{4}(e^{\sqrt{2}\tilde{\alpha}} + e^{-\sqrt{2}\tilde{\alpha}})$$

is an Ising vector in $V_{\sqrt{2}Q(E_7)} = V_{\mathbb{Z}\sqrt{2}\tilde{\alpha}}$.

Structures of $V_{B(nX)}$ and $U_{B(nX)}$. We determine the structures of $V_{B(nX)}$ and $U_{B(nX)}$.

1A case. In this case $\tilde{L}_{1A} \simeq A_1 \oplus E_7$ and we know $V_{B(1A)} \simeq U_{2A}$ by [LYY2]. It follows that $U_{B(1A)} \simeq L(7/10, 0)$ (cf. Section 3.1). It is clear that the weight two subspace of $U_{B(1A)}$ is one-dimensional.

2B case. In this case $\tilde{L}_{2B} \simeq A_1 \oplus A_1 \oplus D_6$ and $E_8/\tilde{L}_{2B} \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Let $\epsilon_1, \dots, \epsilon_8 \in \mathbb{R}^8$ be such that $(\epsilon_i, \epsilon_j) = 2\delta_{i,j}$ for any $i,\, j \in \{1, \dots, 8\}$. Then

$$\sqrt{2}E_8 = \left\{ \sum_{i=1}^{8} a_i\epsilon_i \in \mathbb{R}^8 \;\middle|\; \text{all } a_i \in \mathbb{Z} \text{ or all } a_i \in \mathbb{Z} + \tfrac{1}{2} \text{ and } \textstyle\sum_{i=1}^{8} a_i \equiv 0 \pmod 2 \right\}.$$

It is shown in [LYY2] (see also [DLY2, FLM]) that

$$V_{\sqrt{2}E_8} \simeq V_{\mathcal{K}}^+, \quad \mathcal{K} = \left\{ \sum_{i=1}^{8} a_i\epsilon_i \in \mathbb{R}^8 \;\middle|\; \text{all } a_i \in \mathbb{Z} \text{ or all } a_i \in \mathbb{Z} + \tfrac{1}{2} \right\}. \tag{4.14}$$

By the same argument as in the monstrous 4A-case in [LYY2], we have $V_{B(2B)} \simeq V_{\mathcal{A}}^+$, where $\mathcal{A} = \mathrm{span}_{\mathbb{Z}}\{-\epsilon_1 - \epsilon_2,\, \frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_8)\} \simeq \sqrt{2}A_2$. Thus,

$$V_{B(2B)} \simeq V_{\sqrt{2}A_2}^+. \tag{4.15}$$

There exists a Virasoro frame $L(1/2,0)\otimes L(7/10,0)\otimes L(4/5,0)$ inside $V^+_{\sqrt{2}A_2}$ and the decomposition of $V^+_{\sqrt{2}A_2}$ as a module over the frame is computed in [KMY]. Here we note that $\mathrm{Vir}(\tilde{\omega}_{Q(E_7)}) \simeq L(1/2,0)$ is a member of the Virasoro frame and therefore from (loc. cit.) we obtain

$$U_{B(2B)} \simeq L(7/10,0)\otimes L(4/5,0) \oplus L(7/10,3/5)\otimes L(4/5,7/5). \tag{4.16}$$

By this decomposition, we see that the weight two subspace of $U_{B(2B)}$ is 3-dimensional and therefore coincides with $\mathcal{G}_{B(2B)}$.

Remark 4.5. By the same argument as in [DLY2], one can also show that

$$U_{B(2B)} \simeq L(1/2,0)\otimes V^+_{\mathbb{Z}\gamma} \oplus L(1/2,1/2)\otimes V^+_{\mathbb{Z}\gamma+\gamma/2},$$

where $\langle \gamma,\gamma\rangle = 12$.

2C case. In this case $\tilde{L}_{2C} \simeq A_1 \oplus A_7$ and $E_8/\tilde{L}_{2C} \simeq \mathbb{Z}_4$. It is clear that $V_{B(2C)}$ is isomorphic to the monstrous 4B-algebra U_{4B} discussed in [LYY2]. We know (cf. loc. cit.) that

$$\begin{aligned} U_{4B} \simeq\ & L(1/2,0)\otimes\big[L(7/10,0)\otimes L(7/10,0) \oplus L(7/10,3/2)\otimes L(7/10,3/2)\big] \\ & \oplus L(1/2,1/2)\otimes\big[L(7/10,3/2)\otimes L(7/10,0) \oplus L(7/10,0)\otimes L(7/10,3/2)\big] \end{aligned} \tag{4.17}$$

which gives

$$U_{B(2C)} \simeq L(7/10,0)\otimes L(7/10,0) \oplus L(7/10,3/2)\otimes L(7/10,3/2). \tag{4.18}$$

By this decomposition, we see that the weight two subspace of $U_{B(2C)}$ is 2-dimensional and coincides with $\mathcal{G}_{B(2C)}$.

3A case. In this case $\tilde{L}_{3A} \simeq A_1 \oplus A_2 \oplus A_5$ and $E_8/\tilde{L}_{3A} \simeq \mathbb{Z}_6$. It is clear that $V_{B(3A)}$ is isomorphic to the monstrous 6A-algebra U_{6A} discussed in [LYY2] (see also Appendix A) and $\tilde{\omega}_{Q(E_7)}$ corresponds to the Ising vector $\omega^1 \in U_{6A}$ in Appendix A. The commutant subalgebra $U_{B(3A)} \simeq \mathrm{Com}_{U_{6A}}(\mathrm{Vir}(\omega^1))$ does not have a Virasoro frame which consists of rational unitary Virasoro VOAs. Nevertheless, by [LYY2, Appendix B.3], we have the following decomposition:

$$U_{B(3A)} \simeq \mathcal{W}(4/5)\otimes W_6(0,0) \oplus L(4/5,2/3)^+\otimes W_6(0,4) \oplus L(4/5,2/3)^-\otimes W_6(0,8),$$

where we denoted the $\mathcal{W}(4/5) \simeq L(4/5,0)\oplus L(4/5,3)$-modules as in [HLY] and $W_6(0,0) \simeq \mathrm{Com}_{V_{\sqrt{2}A_5}}(\mathrm{Vir}(\omega_{A_5}-\tilde{\omega}_{A_5}))$ is a W_6-algebra with central charge 5/4 and $W_6(0,4)$, $W_6(0,8)$

are irreducible $W_6(0,0)$-modules see [DLe, DLY3, LYY2] for details.[2] The head characters of $W_6(0,0)$, $W_6(0,4)$ and $W_6(0,8)$ are computed in [LYY2] and it follows that the weight two subspace of $U_{B(3A)}$ is 4-dimensional. Therefore, the Griess algebra of $U_{B(3A)}$ is equal to $\mathcal{G}_{B(3A)}$.

4B case. In this case, $\tilde{L}_{4B} \simeq A_1 \oplus A_1 \oplus A_3 \oplus A_3$ and $E_8/\tilde{L}_{4B} \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_4$. With a similar argument as in the monstrous 4A case of the E_8-observation in [LYY2] and the Baby-monstrous 2B case, we have $V_{B(4B)} \simeq V_{\mathcal{N}}^+$, where $\mathcal{N}$ is a rank 3 lattice defined by $\mathcal{N} = \mathrm{span}_{\mathbb{Z}}\{-\epsilon_1 - \epsilon_2, \frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_8), \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_5 - \epsilon_6 - \epsilon_7 - \epsilon_8)\}$. Here, the ϵ_i, $i = 1, \ldots, 8$, are defined as in the 2B case and the Gram matrix of $\mathcal{N}$ is given by

$$\begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & 1 \\ -2 & 1 & 4 \end{bmatrix}. \tag{4.19}$$

Set

$$\alpha = -\epsilon_1 - \epsilon_2, \quad \beta = \epsilon_3 + \epsilon_4 + \epsilon_5, \quad \gamma = \epsilon_6 + \epsilon_7 + \epsilon_8.$$

Then $\langle \alpha, \alpha \rangle = 4$, $\langle \beta, \beta \rangle = \langle \gamma, \gamma \rangle = 6$ and $\{\alpha, \beta, \gamma\}$ forms an orthogonal frame for $\mathcal{N}$. Moreover, we have

$$\mathcal{N} = K \sqcup (\delta + K),$$

where $K = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \oplus \mathbb{Z}\gamma$ and $\delta = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_8) = \frac{1}{2}(-\alpha + \beta + \gamma)$. Thus, we have

$$\begin{aligned} U_{B(4B)} &= \mathrm{Com}_{V_{\mathcal{N}}^+}(\omega^+(\sqrt{2}\alpha)) \\ &\simeq L(1/2, 0) \otimes \left[V_{\mathbb{Z}\beta}^+ \otimes V_{\mathbb{Z}\gamma}^+ \ \oplus \ V_{\mathbb{Z}\beta}^- \otimes V_{\mathbb{Z}\gamma}^- \right] \\ &\quad \oplus \ L(1/2, 1/2) \otimes \left[V_{\mathbb{Z}\beta+\beta/2}^+ \otimes V_{\mathbb{Z}\gamma+\gamma/2}^- \ \oplus \ V_{\mathbb{Z}\beta+\beta/2}^- \otimes V_{\mathbb{Z}\gamma+\gamma/2}^+ \right]. \end{aligned}$$

The commutant sub-VOA $U_{B(4B)}$ does not have a Virasoro frame which consists of rational unitary Virasoro VOAs. Nevertheless, we see from the decomposition above that the weight two subspace of $U_{B(4B)}$ is 6-dimensional and therefore coincides with $\mathcal{G}_{B(4B)}$.

Remark 4.6. By Lemma 3.3 of [DLY1], we have

$$\ker_{V_{\sqrt{2}A_3}}(\omega_{A_3} - \tilde{\omega}_{A_3})_{(0)} \simeq V_{\sqrt{6}\mathbb{Z}}^+.$$

Since $L_{4B} \simeq A_1 \oplus A_3 \oplus A_3$, we have another proof that $U_{B(4B)}$ contains a full subalgebra of the form

$$L(1/2, 0) \otimes V_{\mathbb{Z}\beta}^+ \otimes V_{\mathbb{Z}\gamma}^+,$$

where $\langle \beta, \beta \rangle = \langle \gamma, \gamma \rangle = 6$.

[2]The labeling of irreducible W_6-modules in [LYY2] is different from the one in [DLY3]. Here we adopt the labeling used in [LYY2]. $W_3(0,0)$ in loc. cit. is called $\mathcal{W}(4/5)$ in the present paper.

Subalgebras generated by $\tilde{f}$ and $\tilde{f}'$. Let

$$\tilde{f} := \tilde{\omega}_{E_7} \quad \text{and} \quad \tilde{f}' := \rho_{nX}\tilde{\omega}_{E_7}. \tag{4.20}$$

By definition, it is clear that $\tilde{f}$ and $\tilde{f}'$ are contained in the Griess subalgebra

$$\mathcal{G}_{B(nX)} = \operatorname{span}_{\mathbb{C}}\{\ \omega^s,\ X^r \mid 1 \le s \le \ell,\ \ 1 \le r \le n-1\}$$

of $U_{B(nX)}$.

Next, we will discuss $\mathcal{G}_{B(nX)}$ and the subalgebra generated by $\tilde{f}$ and $\tilde{f}'$.

1A case. In this case $L_{1A} \simeq E_7$ and ρ_{1A} is trivial. Thus $\tilde{f} = \tilde{f}'$, $\langle \tilde{f}, \tilde{f}' \rangle = 7/20$ and $\tilde{f}$ generates $\mathcal{G}_{B(1A)}$ and $U_{B(1A)}$.

2B case. In this case $L_{2B} \simeq A_1 \oplus D_6$ and $\ell = 2$.

The mutually orthogonal simple Virasoro vectors ω^1 and ω^2 have the central charges 1/2 and 1, respectively, and $X = X^1$ is a highest weight vector for $\mathrm{Vir}(\omega^1) \otimes \mathrm{Vir}(\omega^2)$ with highest weight $(1/2, 3/2)$. By a direct computation, one finds the following commutative algebra structure on $\mathcal{G}_{B(2B)}$:

$a_{(1)}b$	ω^1	ω^2	X
ω^1	$2\omega^1$	0	$\frac{1}{2}X$
ω^2		$2\omega^2$	$\frac{3}{2}X$
X			$128\omega^1 + 192\omega^2$

$\langle a, b\rangle$	ω^1	ω^2	X
ω^1	$\frac{1}{4}$	0	0
ω^2		$\frac{1}{2}$	0
X			64

One also finds that

$$\tilde{f} = \frac{1}{5}\omega^1 + \frac{3}{5}\omega^2 + \frac{1}{20}X, \qquad \tilde{f}' = \frac{1}{5}\omega^1 + \frac{3}{5}\omega^2 - \frac{1}{20}X, \qquad \langle \tilde{f}, \tilde{f}' \rangle = \frac{3}{100},$$

and that the algebra $\mathcal{G}_{B(2B)}$ is generated by $\tilde{f}$ and $\tilde{f}'$. Set

$$u := \frac{4}{5}\omega^1 + \frac{2}{5}\omega^2 - \frac{1}{20}X \quad \text{and} \quad w := -32\omega^1 + 16\omega^2 - X.$$

Then u is a simple $c = 4/5$ Virasoro vector orthogonal to $\tilde{f}$ and w is a highest weight vector for $\mathrm{Vir}(\tilde{f}) \otimes \mathrm{Vir}(u)$ with highest weight $(3/5, 7/5)$. By the fusion rules

$$L(7/10, 3/5) \times L(7/10, 3/5) = L(7/10, 0) + L(7/10, 3/5),$$

$$L(4/5, 7/5) \times L(4/5, 7/5) = L(4/5, 0) + L(4/5, 7/5),$$

one sees that $\tilde{f}$ and $\tilde{f}'$ generate a subalgebra

$$L(7/10, 0) \otimes L(4/5, 0) \ \oplus\ L(7/10, 3/5) \otimes L(4/5, 7/5)$$

which is equal to $U_{B(2B)}$. Note that the decomposition above is not a $\mathbb{Z}_2$-graded extension of $\mathrm{Vir}(\tilde{f}) \otimes \mathrm{Vir}(u)$ since $w_{(1)}w = 768\tilde{f} + 1568u - 24w$.

2C case. In this case $L_{2C} \simeq A_7$ and $\ell = 1$. Then $\omega = \omega^1$ is a Virasoro vector of central charge $7/5$ and $X = X^1$ is a highest weight vector for $\mathrm{Vir}(\omega)$ with highest weight 2. We have the following commutative algebra structure on $\mathcal{G}_{B(2C)}$:

$$\omega_{(1)}X = 2X, \qquad X_{(1)}X = 70X, \qquad \langle \omega, \omega \rangle = 7/10, \qquad \langle X, X \rangle = 70.$$

One verifies that

$$\tilde{f} = \frac{1}{2}\omega + \frac{1}{20}X, \quad \tilde{f}' = \frac{1}{2}\omega - \frac{1}{20}X, \quad \langle \tilde{f}, \tilde{f}' \rangle = 0.$$

Thus, the VOA generated by $\tilde{f}$ and $\tilde{f}'$ is isomorphic to

$$\mathrm{Vir}(\tilde{f}) \otimes \mathrm{Vir}(\tilde{f}') \simeq L(7/10, 0) \otimes L(7/10, 0).$$

In this case, $U_{B(2C)} \simeq L(7/10, 0) \otimes L(7/10, 0) \oplus L(7/10, 3/2) \otimes L(7/10, 3/2)$ is not generated by $\tilde{f}$ and $\tilde{f}'$.

3A case. In this case $L_{3A} \simeq A_2 \oplus A_5$ and $\ell = 2$.

Then ω^1 and ω^2 are mutually orthogonal Virasoro vectors with central charges $4/5$ and $5/4$, respectively, and X^j, $j = 1, 2$, are highest weight vectors for $\mathrm{Vir}(\omega^1) \otimes \mathrm{Vir}(\omega^2)$ with highest weight $(2/3, 4/3)$. We have the following commutative algebra structure on $\mathcal{G}_{B(3A)}$:

$a_{(1)}b$	ω^1	ω^2	X^1	X^2
ω^1	$2\omega^1$	0	$\frac{2}{3}X^1$	$\frac{2}{3}X^2$
ω^2		$2\omega^2$	$\frac{4}{3}X^1$	$\frac{4}{3}X^2$
X^1			$12X^2$	$75\omega^1 + 96\omega^2$
X^2				$12X^1$

$\langle a, b \rangle$	ω^1	ω^2	X^1	X^2
ω^1	$\frac{2}{5}$	0	0	0
ω^2		$\frac{5}{8}$	0	0
X^1			0	45
X^2				0

One verifies that

$$\tilde{f} = \frac{1}{4}\omega^1 + \frac{2}{5}\omega^2 + \frac{1}{20}X^1 + \frac{1}{20}X^2, \quad \tilde{f}' = \frac{1}{4}\omega^1 + \frac{2}{5}\omega^2 + \frac{\zeta_3}{20}X^1 + \frac{\zeta_3^{-1}}{20}X^2, \quad \langle \tilde{f}, \tilde{f}' \rangle = \frac{1}{80}.$$

and checks again that $\tilde{f}$ and $\tilde{f}'$ generate $\mathcal{G}_{B(3A)}$.

4B case. In this case $L_{4B} \simeq A_3 \oplus A_3 \oplus A_1$ and $\ell = 3$.

One gets that ω^1, ω^2 and ω^3 are mutually orthogonal Virasoro vectors with central charges 1, 1 and $1/2$, respectively. The vectors X^1, X^2 and X^3 are highest weight vectors for $\mathrm{Vir}(\omega^1) \otimes \mathrm{Vir}(\omega^2) \otimes \mathrm{Vir}(\omega^3)$ with highest weight $(3/4, 3/4, 1/2)$, $(1, 1, 0)$ and $(3/4, 3/4, 1/2)$, respectively.

We also have the following commutative algebra structure on $\mathcal{G}_{B(4B)}$:

$a_{(1)}b$	ω^1	ω^2	ω^3	X^1	X^2	X^3
ω^1	$2\omega^1$	0	0	$\frac{3}{4}X^1$	X^2	$\frac{3}{4}X^3$
ω^2		$2\omega^2$	0	$\frac{3}{4}X^1$	X^2	$\frac{3}{4}X^3$
ω^3			$2\omega^3$	$\frac{1}{2}X^1$	0	$\frac{1}{2}X^3$
X^1				$8X^2$	$9X^3$	$48(\omega^1+\omega^2)+64\omega^3$
X^2					$72(\omega^1+\omega^2)$	$9X^1$
X^3						$8X^2$

$$\langle \omega^1, \omega^1\rangle = \langle \omega^2, \omega^2\rangle = 1/2, \quad \langle \omega^3, \omega^3\rangle = 1/4, \quad \langle X^1, X^3\rangle = 32, \quad \langle X^2, X^2\rangle = 36.$$

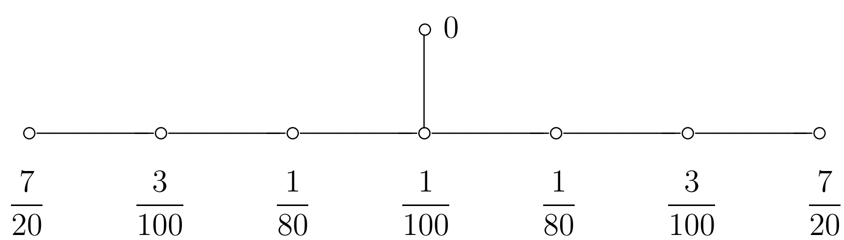
One verifies that

$$\begin{aligned} \tilde{f} &= \frac{3}{10}\omega^1 + \frac{3}{10}\omega^2 + \frac{1}{5}\omega^3 + \frac{1}{20}X^1 + \frac{1}{20}X^2 + \frac{1}{20}X^3, \\ \tilde{f}' &= \frac{3}{10}\omega^1 + \frac{3}{10}\omega^2 + \frac{1}{5}\omega^3 + \frac{\zeta_4}{20}X^1 + \frac{\zeta_4^2}{20}X^2 + \frac{\zeta_4^3}{20}X^3, \end{aligned} \qquad \langle \tilde{f}, \tilde{f}'\rangle = \frac{1}{100}.$$

In this case, the Griess algebra $\mathcal{G}_{B(4B)}$ is not generated by $\tilde{f}$ and $\tilde{f}'$. Denote by ν the nontrivial diagram automorphism of the affine E_7 diagram, that is, ν is defined as: $\alpha_0 \mapsto \alpha_6 \mapsto \alpha_0$, $\alpha_1 \mapsto \alpha_5 \mapsto \alpha_1$, $\alpha_2 \mapsto \alpha_4 \mapsto \alpha_2$, $\alpha_3 \mapsto \alpha_3$ and $\alpha_7 \mapsto \alpha_7$ on the diagram (4.11). Since $\sqrt{2}E_7$ is doubly even, we have a splitting $\mathrm{Aut}(V_{\sqrt{2}E_7}) \simeq \mathrm{Hom}_{\mathbb{Z}}(E_7, \mathbb{C}^*) \rtimes \mathrm{Aut}(E_7)$. Then ν canonically acts on $\mathcal{G}_{B(4B)}$ and we find that $\tilde{f}$ and $\tilde{f}'$ generate the fixed point subalgebra

$$\mathcal{G}_{B(4B)}^{\langle\nu\rangle} = \mathrm{span}_{\mathbb{C}}\{\omega^1+\omega^2, \omega^3, X^1, X^2, X^3\}.$$

Summarizing the computations above, we have the following table of values of inner products between $\tilde{f}$ and $\tilde{f}'$:



Remark 4.7. By the computations above, we find that the Griess subalgebra generated by $\tilde{f}$ and $\tilde{f}'$ coincides with the fixed point subalgebra $\mathcal{G}_{B(nX)}^{\langle\nu\rangle}$ if $nX = 2C$ and $4B$. These are the cases when the corresponding nodes are the fixed points of the diagram automorphism ν.

5 The Baby Monster

In this section, we will discuss the properties of the commutant vertex operator subalgebra $V\mathbb{B}^\natural$ of the Moonshine VOA $V^\natural$. It is known that $\mathrm{Aut}(V\mathbb{B}^\natural) = \mathbb{B}$. We will show that there exists a one-to-correspondence between 2A-involutions of $\mathbb{B}$ and simple $c = 7/10$ Virasoro vectors of σ-type in $V\mathbb{B}^\natural$.

Finally, we will discuss the embedding of $U_{B(nX)}$ into $V\mathbb{B}^\natural$. The main idea is to embed the root lattice E_7 into E_8 and view $U_{B(nX)}$ as a certain commutant subalgebra of the lattice VOA $V_{\sqrt{2}E_8}$. Then we will show that the product of two σ-involutions generated by simple $c = 7/10$ Virasoro vectors in $U_{B(nX)}$ exactly belong to the conjugacy class nX in $\mathbb{B}$. By this procedure, we obtain a VOA description of the E_7 structure inside $\mathbb{B}$.

The automorphism group of the Moonshine VOA $V^\natural$ is the Monster $\mathbb{M}$ [FLM]. Consider the monstrous Griess algebra of dimension 196884 [C, G]. It is known that the monstrous Griess algebra is naturally realized as the subspace of weight 2 of $V^\natural$ [FLM], which we call the Griess algebra of $V^\natural$ and denote by $\mathcal{G}^\natural$. We will freely use the character tables in [ATLAS], although no explicit proofs for their correctness have been published up to now.

5.1 The Baby Monster vertex operator algebra $V\mathbb{B}^\natural$

The following one-to-one correspondence is crucial in the rest of this paper.

Theorem 5.1 ([C, Mi1, Ma1, Hö2])**.** *The map which associates to an Ising vector of $V^\natural$ its τ-involution given in Theorem 2.2 defines a bijection between the set of Ising vectors of $V^\natural$ and the 2A-conjugacy class of the Monster $\mathbb{M} = \mathrm{Aut}(V^\natural)$.*

First, Conway [C] showed that every 2A-element determines a so called axial vector of the Griess algebra which is up to rescaling an Ising vector. Then Miyamoto [Mi1] showed that any Ising vector defines an involutive automorphism of a VOA. In the case of the Moonshine VOA, this recovers the 2A-element [Ma1]. In [Hö2], it was shown that this correspondence is actually one-to-one as remarked in [Mi1].

By Theorem 5.1, we see that the number of Ising vectors of $V^\natural$ equals the number of 2A-involutions of the Monster. As constructed in [FLM, Mi2], the Moonshine VOA $V^\natural$ has a compact real form $V^\natural_{\mathbb{R}}$. The 196883-dimensional irreducible representation of $\mathbb{M}$ is real [ATLAS] and it follows that the Griess algebra structure of the weight 2 subspace of $V^\natural_{\mathbb{R}}$ is isomorphic to the monstrous Griess algebra over the real numbers considered in [C, G]. It is shown in [C] that the number of Ising vectors in $V^\natural_{\mathbb{R}}$ is not less than the number of 2A-involutions of the Monster, and an Ising vector of $V^\natural_{\mathbb{R}}$ is still an Ising vector in the complex Moonshine VOA $V^\natural = \mathbb{C} \otimes_{\mathbb{R}} V^\natural_{\mathbb{R}}$. Therefore, there are no “complex” Ising vectors in $V^\natural$ and we obtain the following.

Proposition 5.2. *Let $V^\natural_{\mathbb{R}}$ be a compact real form of $V^\natural$ as in [FLM, Mi2]. Then every Ising vector of $V^\natural$ is contained in $V^\natural_{\mathbb{R}}$.*

Let us recall the Baby Monster VOA discussed in [Hö1, Hö2, Hö3, Y]. We fix an Ising vector $e \in V^\natural$. The centralizer $C_{\mathbb{M}}(\tau_e)$ is isomorphic to a 2-fold cover $\langle \tau_e \rangle \cdot \mathbb{B}$ of the Baby Monster $\mathbb{B}$ so that the commutant subalgebra

$$V\mathbb{B}^\natural := \mathrm{Com}_{V^\natural}(\mathrm{Vir}(e)) = \ker_{V^\natural} e_{(0)} \tag{5.1}$$

affords a natural action of the Baby Monster. Since all the Ising vectors of $V^\natural$ are mutually conjugate, the VOA $V\mathbb{B}^\natural$ is well-defined up to isomorphism. So we call $V\mathbb{B}^\natural$ the *Baby Monster VOA*[3]. The Baby Monster VOA is a framed VOA and its structure as well as its representation are studied in [Hö1, Hö2, Y]. It is proved in [Hö2] that the Baby Monster $\mathbb{B}$ is the full automorphism group of $V\mathbb{B}^\natural$. (See also [Y] for another proof.)

As obtained in [C, MeN], the $C_{\mathbb{M}}(\tau_e)$-module $\mathcal{G}^\natural$ decomposes as follows:

$$\begin{array}{ccccccccccc} \mathcal{G}^\natural & = & \underline{\mathbf{1}} & \oplus & \underline{\mathbf{1}} & \oplus & \underline{\mathbf{4371}} & \oplus & \underline{\mathbf{96255}} & \oplus & \underline{\mathbf{96256}} \\ e_{(1)} & : & 2 & & 0 & & 1/2 & & 0 & & 1/16 \\ \tau_e & : & +1 & & +1 & & +1 & & +1 & & -1 \end{array} \tag{5.2}$$

The Ising vector e belongs to the first component in the above decomposition, in particular it is contained in $(\mathcal{G}^\natural)^{C_{\mathbb{M}}(\tau_e)}$.

By construction, the Griess algebra of $V\mathbb{B}^\natural$ is of dimension 96256 and has the decomposition $V\mathbb{B}^\natural_2 = \underline{\mathbf{1}} \oplus \underline{\mathbf{96255}}$ as a $\mathbb{B}$-module, where the principal module is spanned by the conformal vector of $V\mathbb{B}^\natural$ with central charge $47/2$.

5.2 The $\{3, 4\}$-transposition property

We fix a 2A-involution t of $\mathbb{B} = \mathrm{Aut}(V\mathbb{B}^\natural)$. Then $C_{\mathbb{B}}(t) = 2{\cdot}({}^2E_6(2)){:}2$ (cf. [ATLAS]) and the Griess algebra of $V\mathbb{B}^\natural$ has the following decomposition into irreducibles as a $C_{\mathbb{B}}(t)$-module:

$$V\mathbb{B}^\natural_2 = \underline{\mathbf{1}} \oplus \underline{\mathbf{1}} \oplus \underline{\mathbf{1938}} \oplus \underline{\mathbf{48620}} \oplus \underline{\mathbf{45696}}. \tag{5.3}$$

By the decomposition above, the Griess algebra of the fixed point subalgebra $(V\mathbb{B}^\natural)^{C_{\mathbb{B}}(t)}$ forms a 2-dimensional semisimple commutative algebra. Below we will argue that the shorter Virasoro vector in $(V\mathbb{B}^\natural)^{C_{\mathbb{B}}(t)}$ has central charge $c = 7/10$ and is of σ-type. Then we will prove that the correspondence between 2A-elements of $\mathbb{B}$ and simple $c = 7/10$ Virasoro vectors of $V\mathbb{B}^\natural$ of σ-type is one-to-one.

[3] Our $V\mathbb{B}^\natural$ is actually the even part of the *shorter Moonshine module* constructed in [Hö1] and it is denoted by $VB^\natural_{(0)}$ in [Hö1].

Let f be a derived $c = 7/10$ Virasoro vector in $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e))$ with respect to e. Let $U \subset V^\natural$ be the corresponding sub-VOA isomorphic to U_{2A}, that is, $e + f$ is a Virasoro frame of U. For an irreducible U-module M, we set the space of multiplicities by

$$H_M := \mathrm{Hom}_{U_{2A}}(M, V^\natural)$$

and we have the decomposition

$$V^\natural = \bigoplus_{M \in \mathrm{Irr}(U_{2A})} M \otimes H_M. \tag{5.4}$$

The space H_M affords a natural action of the commutant subalgebra $\mathrm{Com}_{V^\natural}(U)$. The Virasoro vectors e and f are mutually orthogonal and the subalgebra $\mathrm{Vir}(e) \otimes \mathrm{Vir}(f)$ forms a Virasoro frame of U isomorphic to $L(1/2, 0) \otimes L(7/10, 0)$. We adopt the labeling of irreducible U_{2A}-modules with respect to this frame.

Lemma 5.3. *The top weight $h(H_M)$ and the dimension $d(H_M)$ of the top level of the $\mathrm{Com}_{V^\natural}(U)$-modules H_M are given by the following table:*

M	$U(0,0)$	$U(1/2,0)$, $U(1/16,7/16)^\pm$	$U(0,3/5)$	$U(0,1/10)$, $U(1/16,3/80)^\pm$
$h(H_M)$	0	3/2	7/5	19/10
$d(H_M)$	1	2432	1938	45696

(5.5)

Moreover, the dimension of the Griess algebra of the commutant subalgebra $\mathrm{Com}_{V^\natural}(U) = H_{U(0,0)}$ *is* 48621.

Proof: For $h \in \mathbb{C}$ and $x \in \mathcal{G}^\natural$, we set $\mathcal{G}^\natural_x[h] := \{v \in \mathcal{G}^\natural \mid x_{(1)}v = hv\}$. It is shown in [Hö4] that $\mathcal{G}^\natural$ forms a conformal 11-design[4] and one can use the Matsuo-Norton trace formula obtained in [Ma1] up to five compositions of adjoint actions of elements in $\mathcal{G}^\natural$.

Let $u \in \mathcal{G}^\natural$ be any Ising vector. Applying the formula, the following decomposition is obtained in (loc. cit.):

$$\begin{aligned} \mathcal{G}^\natural &= \mathcal{G}^\natural_u[2] \oplus \mathcal{G}^\natural_u[0] \oplus \mathcal{G}^\natural_u[1/2] \oplus \mathcal{G}^\natural_u[1/16], \\ \dim \mathcal{G}^\natural_u[2] &= 1, \quad \dim \mathcal{G}^\natural_u[0] = \dim \mathcal{G}^\natural_u[1/16] = 96256, \quad \dim \mathcal{G}^\natural_u[1/2] = 4371. \end{aligned} \tag{5.6}$$

Let $v \in \mathcal{G}^\natural$ be a simple $c = 7/10$ Virasoro vector. It is also calculated in (loc. cit.) by a similar argument that

$$\begin{aligned} \mathcal{G}^\natural &= \mathcal{G}^\natural_v[2] \oplus \mathcal{G}^\natural_v[0] \oplus \mathcal{G}^\natural_v[3/2] \oplus \mathcal{G}^\natural_v[1/10] \oplus \mathcal{G}^\natural_v[3/5] \oplus \mathcal{G}^\natural_v[7/16] \oplus \mathcal{G}^\natural_v[3/80], \\ \dim \mathcal{G}^\natural_v[2] &= \dim \mathcal{G}^\natural_v[3/2] = 1, \quad \dim \mathcal{G}^\natural_v[0] = 51054, \quad \dim \mathcal{G}^\natural_v[1/10] = 47634, \\ \dim \mathcal{G}^\natural_v[3/5] &= 1938, \quad \dim \mathcal{G}^\natural_v[7/16] = 4864, \quad \dim \mathcal{G}^\natural_v[3/80] = 91392. \end{aligned} \tag{5.7}$$

[4] Or, one can say the Moonshine VOA is of class $\mathcal{S}^{11}$ as in [Ma1].

This allows to compute the weights and dimensions of the top levels of H_M for $M = U(0,0)$, $U(1/2,0)$, $U(0,3/5)$ and $U(0,1/10)$ as in the Lemma. Since (5.6) and (5.7) hold for arbitrary Ising vectors u and simple $c = 7/10$ Virasoro vectors v of $V^\natural$, it follows from the conjugate relations (3.6), (3.7) and Lemma 3.1 that $d(H_{U(1/2,0)}) = d(H_{U(1/16,7/16)^\pm})$ and $d(H_{U(0,1/10)}) = d(H_{U(1/16,3/80)^\pm})$. Then one obtains the table. ∎

Recall the group homomorphism $\varphi_e : \mathrm{Stab}_{\mathrm{Aut}(V^\natural)}(e) \to \mathrm{Aut}(\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e))) \simeq \mathbb{B}$ defined in (3.11). By the one-to-one correspondence in Theorem 5.1, we see $\mathrm{Stab}_{\mathrm{Aut}(V^\natural)}(e) = C_{\mathbb{M}}(\tau_e) \simeq \langle \tau_e \rangle.\mathbb{B}$ and therefore φ_e is surjective.

By Lemma 3.1, U_{2A} contains three Ising vectors e, e' and e''. As a derived Virasoro vector, f is of σ-type on the commutant $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e))$ by Lemma 3.6 and one has $\varphi_e(\tau_{e'}) = \varphi_e(\tau_{e''}) = \sigma_f$ by Lemma 3.7.

Proposition 5.4. $\varphi_e(\tau_{e'}) = \varphi_e(\tau_{e''}) = \sigma_f$ *defines a 2A-element of the Baby Monster.*

Proof: Let us compute the trace of σ_f on the Griess algebra of $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e))$. As a $\mathrm{Vir}(f) \otimes \mathrm{Com}_V(U_{2A})$-module, we have the following decomposition:

$$\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e)) = [0] \otimes H_{U(0,0)} \oplus [3/2] \otimes H_{U(1/2,0)} \oplus [1/10] \otimes H_{U(0,1/10)} \oplus [3/5] \otimes H_{U(0,3/5)}.$$

By (3.8) and (5.5), we have

$$\begin{aligned}\mathrm{Tr}_{\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e))_2} \sigma_f &= \dim \mathrm{Vir}(f)_2 + \dim(H_{U(0,0)})_2 - d(H_{U(0,1/10)}) + d(H_{U(0,3/5)}) \\ &= 1 + 48621 - 45696 + 1938 = 4864.\end{aligned}$$

By [ATLAS], we see that $\sigma_f = \varphi_e(\tau_{e'})$ has to be a 2A-involution of the Baby Monster. ∎

Remark 5.5. For the computation of $\mathrm{Tr}_{\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e))_2} \sigma_f$, we only need to use the representation theory of U_{2A} and the fact that $V_2^\natural$ forms a conformal 11-design. No information about $\mathrm{Aut}(V^\natural)$ nor $\mathrm{Aut}(V\mathbb{B}^\natural)$ is required for this part.

Lemma 5.6. $\varphi_e^{-1} C_{\mathbb{B}}(\varphi_e(\tau_{e'}))$ *stabilizes the subset* $\{e', e''\}$ *of* $V^\natural$.

Proof: Take any $g \in \varphi_e^{-1} C_{\mathbb{B}}(\varphi_e(\tau_{e'}))$. Since $g\tau_x g^{-1} = \tau_{gx}$ for any Ising vector x, it suffices to show $g\{\tau_{e'}, \tau_{e''}\}g^{-1} = \{\tau_{e'}, \tau_{e''}\}$ by the one-to-one correspondence of Theorem 5.1. We have $\varphi_e(g\{\tau_{e'}, \tau_{e''}\}g^{-1}) = \varphi_e(g)\{\varphi_e(\tau_{e'})\}\varphi_e(g)^{-1} = \{\varphi_e(\tau_{e'})\}$. By Theorem 3.4 we know $\tau_{e''} = \tau_e \tau_{e'}$. Since $\ker \varphi_e = \langle \tau_e \rangle$, we get $g\{\tau_{e'}, \tau_{e''}\}g^{-1} \subset \varphi_e^{-1}(\{\varphi_e(\tau_{e'})\}) = \{\tau_{e'}, \tau_e \tau_{e'}\} = \{\tau_{e'}, \tau_{e''}\}$. ∎

Proposition 5.7. *A derived* $c = 7/10$ *Virasoro vector* f *of* $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e))$ *with respect to* e *is fixed by the centralizer of the 2A-involution* $\sigma_f = \varphi_e(\tau_{e'})$ *of the Baby Monster.*

Proof: By Lemma 3.1, f has the expression

$$f = -\frac{1}{5}e + \frac{4}{5}(e' + e''). \tag{5.8}$$

Therefore, f is fixed by the centralizer of $\varphi_e(\tau_{e'})$ by Lemma 5.6. ∎

Remark 5.8. We can define a compact real form $V\mathbb{B}^\natural_{\mathbb{R}}$ using the compact real form $V^\natural_{\mathbb{R}}$. It follows from Proposition 5.2 that every derived $c = 7/10$ Virasoro vector in $V\mathbb{B}^\natural$ is real, since it is an $\mathbb{R}$-linear combination of real Ising vectors by (5.8).

Now we establish the one-to-one correspondence between 2A-involutions of the Baby Monster and derived $c = 7/10$ Virasoro vectors of $V\mathbb{B}^\natural$.

Theorem 5.9. *The map which associates a derived $c = 7/10$ Virasoro vector to its σ-involution defines a bijection between the set of all derived $c = 7/10$ Virasoro vectors of* $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e))$ *with respect to e and the 2A-conjugacy class of the Baby Monster* $\mathbb{B} = \mathrm{Aut}(\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e)))$.

Proof: The map of the theorem is equivariant with respect to the natural action of $\varphi_e(\mathrm{Stab}_{\mathrm{Aut}(V^\natural)}(e))$ on the derived vectors and the conjugation action of $\mathbb{B}$ on the set of its 2A-involutions, respectively.

The U_{2A}-algebra can be embedded into $V^\natural$ since $U_{2A} \subset V^+_{\sqrt{2}E_8}$. Because the action of $\mathrm{Aut}(V^\natural)$ on the Ising vectors of $V^\natural$ is transitive by Theorem 5.1, we may assume that e is contained in the chosen embedding of U_{2A}. The vector $f = \omega_{U_{2A}} - e$ is a derived $c = 7/10$ Virasoro vector of $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e))$ with respect to e. The transitivity of the conjugation action of $\mathbb{B}$ on the 2A-involutions shows the surjectivity of the map of the theorem.

For the injectivity, we fix a 2A-involution t of $\mathbb{B} = \mathrm{Aut}(\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e)))$. By Proposition 5.7, any derived $c = 7/10$ Virasoro vector f in $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e))$ such that $\sigma_f = t$ is contained in $(V\mathbb{B}^\natural)^{C_{\mathbb{B}}(t)}$. We have seen in (5.3) that the fixed point subalgebra $(V\mathbb{B}^\natural_2)^{C_{\mathbb{B}}(t)}$ is spanned by two mutually orthogonal Virasoro vectors. Thus f is the unique shorter Virasoro vector of $(V\mathbb{B}^\natural)^{C_{\mathbb{B}}(t)}$ of central charge 7/10. ∎

Let t be a 2A-involution of the Baby Monster. By the arguments above we have determined that the unique shorter Virasoro vector of $(V\mathbb{B}^\natural)^{C_{\mathbb{B}}(t)}$ has central charge 7/10.

Corollary 5.10. *Every 2A-involution t of the Baby Monster* $\mathbb{B} = \mathrm{Aut}(V\mathbb{B}^\natural)$ *uniquely defines a simple $c = 7/10$ Virasoro vector of the fixed point subalgebra* $(V\mathbb{B}^\natural)^{C_{\mathbb{B}}(t)}$.

By Proposition 5.2, we can apply Proposition 3.11 to the Moonshine VOA $V^\natural$. It is shown in Proposition 5.4 that the σ-involutions associated to derived $c = 7/10$ Virasoro vectors of $V\mathbb{B}^\natural$ are in the 2A-conjugacy class of the Baby Monster. Therefore, we have recovered the famous $\{3, 4\}$-transposition property of the Baby Monster from the view point of vertex operator algebras.

Corollary 5.11. *The 2A involutions of the Baby Monster satisfy the* $\{3,4\}$*-transposition property.*

Remark 5.12. Although we used the character table to identify 2A-involutions of the Baby Monster with σ-involutions associated to derived $c = 7/10$ Virasoro vectors in Proposition 5.4, it is worth to note that the proof of Proposition 3.11 uses only the theory of vertex operator algebras. Hence the result that the product of two σ-involutions associated to derived $c = 7/10$ Virasoro vectors in $V\mathbb{B}^\natural$ is bounded by 4 follows from the theory of vertex operator algebras.

We finally show that every simple $c = 7/10$ Virasoro vector in $V\mathbb{B}^\natural$ of σ-type is in fact a derived vector with respect to e.

Theorem 5.13. *The map which associates to a simple* $c = 7/10$ *Virasoro vector of* $V\mathbb{B}^\natural$ *of* σ*-type its* σ*-involution defines a bijection between the set of simple* $c = 7/10$ *Virasoro vectors of* $V\mathbb{B}^\natural$ *of* σ*-type and the 2A-conjugacy class of the Baby Monster* $\mathbb{B} = \mathrm{Aut}(V\mathbb{B}^\natural)$.

Proof: We identify $V\mathbb{B}^\natural$ with the subalgebra $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e))$ of $V^\natural$. Let v be a simple $c = 7/10$ Virasoro vector of $V\mathbb{B}^\natural$ of σ-type. Then, a possible eigenvalue of $v_{(1)}$ on the Griess algebra of $V\mathbb{B}^\natural$ is one of 2, 0, 3/5, 1/10 and 3/2 by Lemma 2.6. We set $d_v(h) := \dim V\mathbb{B}^\natural_v[h] \cap V\mathbb{B}^\natural_2$ for $h \in \{0, {}^3\!/_5, {}^1\!/_{10}, {}^3\!/_2\}$. It is shown in [Hö4] that the Griess algebra of $V\mathbb{B}^\natural$ forms a conformal 7-design[5] and we can apply the Matsuo-Norton trace formula in [Ma1] to compute the trace of the adjoint action on the Griess algebra of $V\mathbb{B}^\natural$ for up to three compositions. As a result, we find the following unique solution:

$$d_v(0) = 48621, \quad d_v({}^3\!/_5) = 1938, \quad d_v({}^1\!/_{10}) = 45696, \quad d_v({}^3\!/_2) = 0.$$

The trace of the σ-involution σ_v is equal to $1 + 48621 + 1938 - 45696 = 4864$ and we see that σ_v must belong to the 2A-conjugacy class of the Baby Monster by [ATLAS]. Therefore, every simple $c = 7/10$ Virasoro vector of σ-type defines a 2A-involution of the Baby Monster.

It is easy to see that $\sigma_{gv} = g\sigma_v g^{-1}$ for any $g \in \mathrm{Aut}(V\mathbb{B}^\natural)$, i.e., the assignment is equivariant. Similar as discussed in the proof of Theorem 5.9 it follows that the assignment is surjective.

It remains to show the injectivity. The idea of the following argument is taken from [Hö2]. Let v be a simple $c = 7/10$ Virasoro vector of $V\mathbb{B}^\natural$ of σ-type. Then σ_v is a 2A-element of the Baby Monster and there exists by Theorem 5.9 a unique derived $c = 7/10$ Virasoro vector $f \in (V\mathbb{B}^\natural)_2^{C_{\mathbb{B}}(\sigma_v)}$ such that $\sigma_v = \sigma_f$. We will prove that $v = f$.

[5] It is shown in [Hö3] that $V\mathbb{B}^\natural$ is also of class $\mathcal{S}^7$ in the sense of Matsuo [Ma1].

Let $V\mathbb{B}^\natural_2 = X^+ \oplus X^-$ be the eigenspace decomposition such that σ_f acts by ± 1 on $X^\pm$. Then $X^+ = \underline{\mathbf{1}} \oplus \underline{\mathbf{1}} \oplus \underline{\mathbf{1938}} \oplus \underline{\mathbf{48620}}$ and $X^- = \underline{\mathbf{45696}}$ as $C_{\mathbb{B}}(\sigma_f)$-modules by (5.3). It is clear that $v \in X^+$ and σ_v also acts by ± 1 on $X^\pm$. For $\varrho \in \mathrm{End}(X^-)$ and $k \in C_{\mathbb{B}}(\sigma_v)$, we define ${}^k\varrho := k\varrho k^{-1}$. Then $\mathrm{End}(X^-)$ becomes a left $C_{\mathbb{B}}(\sigma_v)$-module. Define $\mu : X^+ \to \mathrm{End}(X^-)$ by $\mu(a)x = a_{(1)}x$ for $a \in X^+$ and $x \in X^-$. Then for $k \in C_{\mathbb{B}}(\sigma_f)$ we have

$$\mu(ka)x = (ka)_{(1)}x = (ka)_{(1)}(kk^{-1}x) = k\left(a_{(1)}(k^{-1}x)\right) = k\mu(a)k^{-1}x = ({}^k\mu)(a)x$$

so that μ is a $C_{\mathbb{B}}(\sigma_f)$-homomorphism. Let us consider the $C_{\mathbb{B}}(\sigma_f)$-submodule $\mu(X^+)$. As we will see explicitly in Section 5.3, the 1938 and 48620-dimensional components of X^+ act non-trivially on X^- via μ. (See Remark 5.19 below.) The $(1+1)$-dimensional component is spanned by the simple $c = 7/10$ Virasoro vector f and the conformal vector ω of $V\mathbb{B}^\natural$, and $\mu(f)$ and $\mu(\omega)$ act on X^- by scalars $1/10$ and 2, respectively. Therefore, $\ker_{X^+}\mu = \underline{\mathbf{1}}$ and $\mu(X^+) = \underline{\mathbf{1}} \oplus \underline{\mathbf{1938}} \oplus \underline{\mathbf{48620}}$ as $C_{\mathbb{B}}(\sigma_f)$-modules. Since $\mu(v)$ also acts by $1/10$ on X^-, we see that $f - v \in \ker_{X^+}\mu$. Hence, $v \in (V\mathbb{B}^\natural)^{C_{\mathbb{B}}(\sigma_f)}$, and it follows $v = f$ since f is the unique simple $c = 7/10$ Virasoro vector in $(V\mathbb{B}^\natural)^{C_{\mathbb{B}}(\sigma_f)}$. This shows that the association $v \mapsto \sigma_v$ is injective and we have established the desired bijection. ∎

Corollary 5.14. *Every simple $c = 7/10$ Virasoro vector of $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(e))$ of σ-type is also a derived Virasoro vector with respect to $e \in V^\natural$.*

We have seen in Lemma 3.6 that a derived $c = 7/10$ Virasoro vector is of σ-type and therefore both notions coincide in the case of $V\mathbb{B}^\natural$.

5.3 Embedding of $U_{B(nX)}$ into $V\mathbb{B}^\natural$

Consider the E_8-lattice. We fix an embedding $E_7 \subset E_8$. Then $Q(E_7) = \mathrm{Ann}_{E_8}(E_7) \simeq A_1$ and we have a full rank sublattice $Q(E_7) \oplus E_7 \simeq A_1 \oplus E_7$ of E_8. Since the index $[E_8 : A_1 \oplus E_7]$ is 2, we have a coset decomposition

$$E_8 = A_1 \oplus E_7 \sqcup (\delta + A_1 \oplus E_7)$$

with some $\delta \in E_8$. Correspondingly, we obtain a decomposition

$$V_{\sqrt{2}E_8} = V_{\sqrt{2}(A_1 \oplus E_7)} \oplus V_{\sqrt{2}(\delta + A_1 \oplus E_7)}.$$

Define $\eta \in \mathrm{Aut}(V_{\sqrt{2}E_8})$ by

$$\eta = \mathrm{id} \quad \text{on} \quad V_{\sqrt{2}(A_1 \oplus E_7)}, \qquad \eta = -1 \quad \text{on} \quad V_{\sqrt{2}(\delta + A_1 \oplus E_7)}.$$

Then η is clearly in $\mathcal{F}_{nX}$, where $\mathcal{F}_{nX}$ is the renamed object $\mathcal{F}(i)$ which is defined in (4.9). Indeed, $\mathcal{F}_{nX}$ is generated by η and ρ_{nX}. Note that we can write down ρ_{nX} in exponential form

$$\rho_{nX} = \exp(2\pi\sqrt{-1}\,\gamma^{nX}_{(0)}/n) \quad \text{with suitable } \gamma^{nX} \in L^*_{nX},$$

which also defines an automorphism of $V_{\sqrt{2}E_8}$ and fixes $V_{\sqrt{2}\tilde{L}_{nX}}$ pointwisely.

Remark 5.15. Recall that $\tilde{\omega}_{Q(E_7)}$ defined as in (4.3) with $Q(E_7) \simeq A_1$ is an Ising vector and $U_{B(nX)}$ equals the commutant subalgebra $\mathrm{Com}_{V_{B(nX)}}(\mathrm{Vir}(\tilde{\omega}_{Q(E_7)}))$ (cf. Remark 4.1). Note also that ρ_{nX} fixes $\tilde{\omega}_{Q(E_7)}$.

Now let $U_1 = \langle \hat{e}, \tilde{\omega}_{Q(E_7)} \rangle$ be the subalgebra generated by $\hat{e}$ (cf. (4.10) and $\tilde{\omega}_{Q(E_7)}$ and $U_2 = \rho_{nx}(U_1)$. Then $U_1 \simeq U_2 \simeq U_{2A}$ and we also have

$$\begin{aligned}\mathrm{Vir}(\tilde{f}) &= \mathrm{Com}_{U_1}(\mathrm{Vir}(\tilde{\omega}_{Q(E_7)})), \text{ and} \\ \mathrm{Vir}(\rho_{nX}\tilde{f}) &= \mathrm{Com}_{U_2}(\mathrm{Vir}(\rho_{nx}(\tilde{\omega}_{Q(E_7)}))) = \mathrm{Com}_{U_2}(\mathrm{Vir}(\tilde{\omega}_{Q(E_7)})).\end{aligned}$$

Therefore, the simple $c = 7/10$ Virasoro vectors $\tilde{f} = \tilde{\omega}_{E_7}$ and $\tilde{f}' = \rho_{nX}\tilde{f}$ of $U_{B(nX)}$ are derived Virasoro vectors with respect to $\tilde{\omega}_{Q(E_7)}$.

Proposition 5.16. *For any $nX = 1A$, $2B$, $2C$, $3A$, or $4B$, the VOA $V_{B(nX)}$ can be embedded into the Moonshine VOA $V^\natural$.*

Proof: First, we will note that $V_{B(1A)} \simeq U_{2A}$, $V_{B(2C)} \simeq U_{4B}$, $V_{B(3A)} \simeq U_{6A}$. Since it was shown in [LM] that U_{2A}, U_{4B} and U_{6A} can be embedded into $V^\natural$, we have $V_{B(nX)} \subset V^\natural$ if $nX = $ 1A, 2C, 3A. Moreover, $V_{B(2B)} \simeq V^+_{\sqrt{2}A_2}$. Hence $V_{B(2B)}$ is also contained in $V^+_\Lambda \subset V^\natural$ since $\sqrt{2}A_2$ is clearly contained in the Leech lattice Λ.

Recall that the Leech lattice Λ is generated by elements of the form [CS, Chapter 4] $e_i \pm e_j$ with i, $j \in \Omega$, $\frac{1}{4}e_\Omega - e_1$, and $\frac{1}{2}e_X$, where $e_i = \frac{1}{\sqrt{8}}(0,\dots,4,\dots,0)$, $e_X = \sum_{i\in X} e_i$ for X a vector in the Golay code G_{24}, and $\Omega = \{1,\dots,24\}$. Thus, Λ contains a sublattice isomorphic to $\mathcal{N}$, for example, the sublattice generated by

$$\frac{1}{\sqrt{8}}(4,0^7,-4,0^7,0^8), \quad \frac{1}{\sqrt{8}}(0^8,4,0^7,-4,0^7), \quad \frac{1}{\sqrt{8}}(-2^4,0^4,2^4,0^4,0^8).$$

Here, we will arrange the coordinates such that $(1^4,0^4,1^4,0^4,0^4,0^4) \in G_{24}$. Therefore, $V_{B(4B)} \simeq V^+_{\mathcal{N}}$ is also contained in $V^\natural$. ∎

Remark 5.17. In [GL1] (see also [GL2]), the possible configurations for a pair of $\sqrt{2}E_8$ sublattices (M, N) in a rootless integral lattice have been determined. There are exactly 11 such configurations and 10 of them can be embedded into the Leech lattice. By using these embeddings, one can also obtain explicit embeddings of U_{nX}, $nX = 1A, 2A, \dots, 3C$, into $V^+_\Lambda \subset V^\natural$.

Theorem 5.18. *Let e be an Ising vector in $V^\natural$. Then for any $nX = 1A$, $2B$, $2C$, $3A$, or $4B$, the VOA $U_{B(nX)}$ can be embedded into $V\mathbb{B}^\natural = \mathrm{Com}_{V^\natural}(\mathrm{Vir}(e))$. Moreover, $\sigma_{\tilde{f}}\sigma_{\tilde{f}'}$ belongs to the class nX of $\mathbb{B} = \mathrm{Aut}(V\mathbb{B}^\natural)$.*

Proof: By Proposition 5.16, we can embed $V_{B(nX)}$ into $V^\natural$. Since all Ising vectors of $V^\natural$ are conjugate under the action of $\mathrm{Aut}(V^\natural)$, we may identify $\tilde{\omega}_{Q(E_7)}$ with e. Thus, we have

$$U_{B(nX)} = \mathrm{Com}_{V_{nX}}(\mathrm{Vir}(\tilde{\omega}_{Q(E_7)})) \subset \mathrm{Com}_{V^\natural}(\mathrm{Vir}(e)) = V\mathbb{B}^\natural$$

as desired.

Set $h := \sigma_{\tilde{f}}\sigma_{\tilde{f}'}$. We will show that h belongs to the class nX of $\mathbb{B}$. Recall that there is an exact sequence

$$1 \longrightarrow \langle \tau_e \rangle \longrightarrow C_{\mathbb{M}}(\tau_e) \longrightarrow \mathrm{Aut}(V\mathbb{B}^\natural) \simeq \mathbb{B} \longrightarrow 1$$

with the projection map $\varphi_e : C_{\mathbb{M}}(\tau_e) \to \mathrm{Aut}(V\mathbb{B}^\natural)$. Let e^1 and e^2 be Ising vectors in $V_{B(nX)}$ such that $\varphi_e(\tau_{e^1}) = \sigma_{\tilde{f}}$ and $\varphi_e(\tau_{e^2}) = \sigma_{\tilde{f}'}$. Set $g = \tau_{e^1}\tau_{e^2}$. Then $h = \varphi_e(g)$ and the inverse image $\varphi_e^{-1}(\langle h \rangle)$ has order $2n$ and is generated by τ_e and g.

1A case: In this case, $\tilde{f} = \tilde{f}'$ and hence $h = \sigma_{\tilde{f}}\sigma_{\tilde{f}'}$ belongs to the class 1A.

2B case: In this case, $V_{B(2B)} \simeq V^+_{\sqrt{2}A_2}$. Recall that $V^+_{\sqrt{2}A_2}$ has exactly 6 Ising vectors (cf. [KMY, LSY]). The group generated by the corresponding τ-involutions is an elementary abelian group of order 8, which is not 2A-pure in $\mathbb{M}$ since there exists a pair of mutually orthogonal Ising vectors (cf. Lemma 3.9 and Remark 3.10). Therefore, $g = \tau_{e^1}\tau_{e^2}$ has order 2 and the group generated by τ_e and g is a Klein's 4-group, which is not 2A-pure in $\mathbb{M}$. By using [ATLAS], it is easy to show that $h = \varphi_e(g)$ belongs to the conjugacy class 2B of $\mathbb{B}$.

3A case: In this case, $V_{B(3A)} \simeq U_{6A}$ and the group generated by $\tau_e, \tau_{e_1}, \tau_{e_2}$ is a dihedral group of order 12 and τ_e is in the center. Thus, $g = \tau_{e^1}\tau_{e^2}$ has order 3 or 6. The group generated by τ_e and g is then a cyclic group of order 6 which is generated by a 6A-element of $\mathbb{M}$. By using [ATLAS], one can deduce that $h = \varphi_e(g)$ belongs to the conjugacy class 3A of $\mathbb{B}$.

4B case: In this case, τ_e, τ_{e^1} and τ_{e^2} generate a subgroup isomorphic to a direct product of a cyclic group of order 2 and a dihedral group of order 8 with τ_e in the center. By the sublattice structure

$$E_8 \supset D_5 \oplus A_3 \supset A_1 \oplus A_1 \oplus A_3 \oplus A_3 \simeq \tilde{L}_{4B},$$

it is clear that $U_{4A} \subset V_{B(4B)}$. Thus, e^1, e^2 generate a sub-VOA isomorphic to U_{4A} and $g = \tau_{e^1}\tau_{e^2}$ belongs to the conjugacy class 4A of $\mathbb{M}$. Hence, by using [ATLAS], one can deduce that $h = \varphi_e(g)$ belongs to the conjugacy class 4B of $\mathbb{B}$.

2C case: In this case, $V_{B(2C)} \simeq U_{4B}$. The group generated by τ_e and g is a cyclic group of order 4, which is generated by a 4B-element of $\mathbb{M}$. Now by using [ATLAS], one can deduce that $h = \varphi_e(g)$ belongs to the conjugacy class 2C of $\mathbb{B}$. ∎

Remark 5.19. By Theorem 5.18, we can verify that the 1938 and 48620-dimensional components of X^+ discussed in the proof of Theorem 5.13 act on X^- non-trivially. Here we recover the convention as in the proof of Theorem 5.13. The simple $c = 7/10$ Virasoro vector f acts on the 1938 and 48620-dimensional components by $3/5$ and 0, respectively, and acts on X^- by $1/10$. Thanks to Theorem 5.18, we can identify f with $\tilde{f} \in U_{B(4B)}$, up to conjugacy. In the Griess algebra $\mathcal{G}_{B(4B)}$, $\tilde{f}$ has a 2-dimensional eigenspace for the eigenvalue 0, a 1-dimensional eigenspace for the eigenvalue $3/5$ and a 1-dimensional eigenspace for the eigenvalue $1/10$. One can directly check that these eigenspaces act non-trivially on the $1/10$-eigenspace. Therefore, by the embedding in Theorem 5.18, we see that the corresponding eigensubspaces of X^+ act non-trivially on X^-.

A Appendix: The 6A-algebra

In this Appendix we describe the 6A-algebra U_{6A} explicitly.

Let e^0, e^1 be Ising vectors of $V^\natural$ such that $|\tau_{e^0}\tau_{e^1}| = 6$. Then it is known [ATLAS] that $\tau_{e^0}\tau_{e^1}$ belongs to the 6A-conjugacy class of the Monster.

Let U_{6A} be the subalgebra generated by e^0 and e^1. The subalgebra U_{6A} is called the *6A-algebra* for the Monster and its structure is well-studied in [LYY2]. The Griess algebra of U_{6A} is 8-dimensional and its structure with respect to a specific linear basis $\{\omega^1, \omega^2, \omega^3, X^1, X^2, X^3, X^4, X^5\}$ is as follows (cf. [LYY2], see also [C]):

$a_{(1)}b$	ω^1	ω^2	ω^3	X^1	X^2	X^3	X^4	X^5
ω^1	$2\omega^1$	0	0	$\frac{1}{2}X^1$	0	$\frac{1}{2}X^3$	0	$\frac{1}{2}X^5$
ω^2		$2\omega^2$	0	$\frac{2}{3}X^1$	$\frac{2}{3}X^2$	0	$\frac{2}{3}X^4$	$\frac{2}{3}X^5$
ω^3			$2\omega^3$	$\frac{5}{6}X^1$	$\frac{4}{3}X^2$	$\frac{3}{2}X^3$	$\frac{4}{3}X^4$	$\frac{5}{6}X^5$
X^1				$8X^2$	$9X^3$	$8X^4$	$10X^5$	$72\omega^1 + 60\omega^2 + 48\omega^3$
X^2					$12X^4$	$10X^5$	$75\omega^2 + 96\omega^3$	$10X^1$
X^3						$80\omega^1 + 96\omega^3$	$10X^1$	$8X^2$
X^4							$12X^2$	$9X^3$
X^5								$8X^4$

Thus, ω^i, $i = 1, 2, 3$, are mutually orthogonal Virasoro vectors. The non-trivial linear pairings between these vectors are

$$\langle \omega^1, \omega^1 \rangle = 1/4, \qquad \langle \omega^2, \omega^2 \rangle = 2/5, \qquad \langle \omega^3, \omega^3 \rangle = 5/8,$$

$$\langle X^1, X^5 \rangle = 36, \qquad \langle X^2, X^4 \rangle = 45, \qquad \langle X^3, X^3 \rangle = 40.$$

The conformal vector of U_{6A} is given by $\omega = \omega^1 + \omega^2 + \omega^3$. We can define a canonical $\mathbb{Z}_6$-symmetry as follows.

$$\zeta : \omega^i \longmapsto \omega^i, \quad 1 \le i \le 3, \quad X^j \longmapsto e^{\pi j \sqrt{-1}/3} X^j, \quad 1 \le j \le 5.$$

Lemma A.1. *There are exactly seven Ising vectors in* U_{6A}*, namely,* ω^1 *and* $e^j := \zeta^j e^0$*,* $0 \le j \le 5$*, where*

$$e^0 = \frac{1}{8}\omega^1 + \frac{5}{32}\omega^2 + \frac{1}{4}\omega^3 + \frac{1}{32}(X^1 + X^2 + X^3 + X^4 + X^5).$$

Moreover, the inner products among these Ising vectors are

$$\langle \omega^1, e^i \rangle = \frac{1}{32}, \qquad \langle e^i, e^j \rangle = \begin{cases} 5/2^{10}, & \text{if } i - j \equiv \pm 1 \pmod 6, \\ 13/2^{10}, & \text{if } i - j \equiv \pm 2 \pmod 6, \\ 1/32, & \text{if } i - j \equiv 3 \pmod 6. \end{cases} \tag{A.1}$$

By a straight forward computation, we see that $\omega^1 \in U_{6A}^{\langle \tau_{e^0}, \tau_{e^1} \rangle}$ and we have the following permutation representation

$$\tau_{e^0} = (15)(24), \qquad \tau_{e^1} = (02)(35) \tag{A.2}$$

on $\{e^j \mid 0 \le j \le 5\} \subset U_{6A}$. From this we also find that $\tau_{e^0}\tau_{e^1}$ coincides with ζ on U_{6A}. We also calculate that

$$\omega^1 = e^i + e^{i+3} - 4e^i_{(1)}e^{i+3}, \qquad e^i = \omega^1 + e^{i+3} - 4\omega^1_{(1)}e^{i+3}, \qquad 0 \le i \le 5. \tag{A.3}$$

From this we also see that the subalgebra generated by $\{\omega^1, e^i, e^{i+3}\}$ is isomorphic to the 2A-algebra U_{2A} discussed in Section 3.1. Therefore, by Theorem 3.4, we have

$$\tau_{\omega^1}\tau_{e^i} = \tau_{e^{i+3}}, \qquad \tau_{e^i}\tau_{e^{i+3}} = \tau_{\omega^1}. \tag{A.4}$$

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