McKay's E_6 observation on the largest Fischer group

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Abstract

In this paper, we study McKay's E_6-observation on the largest Fischer 3-transposition group Fi_{24}. We investigate a vertex operator algebra $V\!F^\natural$ of central charge $23\frac{1}{5}$ on which the Fischer group Fi_{24} naturally acts. We show that there is a natural correspondence between dihedral subgroups of Fi_{24} and certain vertex operator subalgebras constructed by the nodes of the affine E_6 diagram by investigating so called derived Virasoro vectors of central charge $6/7$. This allows us to reinterpret McKay's E_6-observation via the theory of vertex operator algebras.

It is also shown that the product of two non-commuting Miyamoto involutions of σ-type associated to derived $c = 6/7$ Virasoro vectors is an element of order 3, under certain general hypotheses on the vertex operator algebra. For the case of $V\!F^\natural$, we identify these involutions with the 3-transpositions of the Fischer group Fi_{24}.

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Contents

1 Introduction

This article is a continuation of our previous work [LM, LYY1, HLY] to give a vertex operator algebra (VOA) theoretical interpretation of McKay's intriguing observations that relate the Monster, the Baby Monster and the largest Fischer 3-transposition group to the affine E_8, E_7 and E_6 Dynkin diagrams. In this article, we will study the E_6-observation recalled below. Our approach here is similar to [HLY], in which the E_7-observation is studied, but other vertex operator algebras are involved and many technical details are different.

The largest Fischer group. The largest Fischer group Fi_{24} was discovered by B. Fischer [F] as a group of order $2^{22} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$ containing a conjugacy class of $306,936$ involutions, which satisfy the 3-*transposition property,* i.e., any non-commuting pair has product of order 3. The group Fi_{24} contains as subgroup of index 2 the derived group Fi_{24}' which is the third largest of the 26 sporadic groups. An extension $3.\mathrm{Fi}_{24}$ of the Fischer group Fi_{24} by a cyclic group of order 3 is the normalizer of a 3A-element of the Monster, the largest of the sporadic groups. In fact, one can construct Fi_{24} from the Monster [G1] and derive its 3-transposition property. One of the main motivation of this article is to study the Fischer group Fi_{24} and to understand the E_6 case of McKay's observations by using the theory of vertex operator algebra.

There exists a class of vertex operator algebras which is closely related to 3-transposition groups [G3, KM, Mi1, Ma2]. Let V be vertex operator algebra which has a simple $c = 1/2$ Virasoro vector e of *σ-type*, that is $V = V_e[0] \oplus V_e[1/2]$ where $V_e[h]$ denotes the sum of irreducible $\mathrm{Vir}(e)$-submodules of V isomorphic to the irreducible highest weight representation $L(1/2, h)$ of the $c = 1/2$ Virasoro algebra with highest weight h (cf. Section 2). Then one can define an involutive automorphism

$$\sigma_e = \begin{cases} 1 & \text{on } V_e[0], \\ -1 & \text{on } V_e[1/2] \end{cases}$$

usually called a σ-involution if $\sigma_e \neq \mathrm{id}_V$ [Mi1]. It was shown by Miyamoto [Mi1] that if the weight one subspace of a VOA is trivial, then a collection of involutions associated to $c = 1/2$ Virasoro vectors of σ-type generates a 3-transposition group. Many interesting examples of 3-transposition groups obtained by σ-type $c = 1/2$ Virasoro vectors have been studied in [G3, KM] and the complete classification is established in [Ma2]. According to [Ma2], all 3-transposition groups realized by σ-type $c = 1/2$ Virasoro vectors are so-called *symplectic type* (cf. [CH2]), and as a result, the Fischer 3-transposition group cannot be obtained by $c = 1/2$ Virasoro vectors of σ-type.

Another result on $c = 1/2$ Virasoro vectors was obtained in [S] where it was shown that the so-called τ-involutions (cf. Theorem 2.2) associated to such vectors generate a 6-transposition group provided the weight one subspace of the VOA is trivial.

In this paper, we introduce a new idea to obtain 3-transposition groups as automorphism groups of vertex operator algebras. We use the so-called 3A-algebra for the Monster (see [Mi3, LYY2, SY] and Section 4) and consider derived $c = 6/7$ Virasoro vectors to define involutive automorphisms of vertex operator algebras. We will show that a collection of involutions associated to derived $c = 6/7$ Virasoro vectors generates a 3-transposition group. The advantage of our method is that we can realize the largest Fischer 3-transposition group as an automorphism subgroup of a special vertex operator algebra $V\!F^\natural$ explained below. This result enables us to study the Fischer 3-transposition groups via the theory of vertex operator algebras.

The Fischer group VOA $V\!F^\natural$. We will investigate a certain vertex operator algebra $V\!F^\natural$ of central charge $23\frac{1}{5}$ on which the Fischer group Fi_{24} naturally acts.

The Monster is the automorphism group of the Moonshine vertex operator algebra $V^\natural$ (cf. [FLM, B]). Let g be a 3A-element of the Monster $\mathbb{M}$. Then the normalizer $N_\mathbb{M}(\langle g\rangle)$ is isomorphic to $3.\mathrm{Fi}_{24}$ and acts on $V^\natural$. A character theoretical consideration in [C, MeN] indicates that the centralizer $C_\mathbb{M}(g) \simeq 3.\mathrm{Fi}_{24}'$ fixes a unique $c = 4/5$ Virasoro vector in $V^\natural$. We will show in Theorem 5.1 that $C_\mathbb{M}(g)$ actually fixes a unique $c = 4/5$ extended Virasoro vertex operator algebra $\mathcal{W} \simeq \mathcal{W}(4/5) = L(4/5, 0) \oplus L(4/5, 3)$ in $V^\natural$. Let

$$VF^\natural = \mathrm{Com}_{V^\natural}(\mathcal{W}),$$

where $\mathrm{Com}_V(U)$ denotes the commutant subalgebra of U in V (see (3.1) and (5.4) for the precise definition), and we call $VF^\natural$ the *Fischer group VOA*. A simple observation shows that $N_\mathbb{M}(\langle g\rangle)$ acts naturally on $VF^\natural = \mathrm{Com}_{V^\natural}(\mathcal{W})$. In fact, we will show that the Fischer group Fi_{24} can be realized as a subgroup of $\mathrm{Aut}(VF^\natural)$.

Main Theorem 1 (Theorem 5.5)**.** *The automorphism group* $\mathrm{Aut}(VF^\natural)$ *of* $VF^\natural$ *contains* Fi_{24} *as a subgroup. Moreover, let* $\mathcal{X}$ *be the full-subalgebra of* $VF^\natural$ *generated by its weight* 2 *subspace. Then* $\mathrm{Aut}(\mathcal{X}) \simeq \mathrm{Fi}_{24}$.

Although we only show that Fi_{24} equals the automorphism group of the Griess algebra of $VF^\natural$, we expect that $\mathrm{Aut}(VF^\natural)$ is exactly Fi_{24} and therefore $VF^\natural$ would provide a VOA model for studying the Fischer group Fi_{24}.

Since $N_\mathbb{M}(\langle g\rangle) \simeq 3.\mathrm{Fi}_{24}$ for any 3A-element g of the Monster, the study of the 3-transpositions in Fi_{24} leads to the study of dihedral subalgebras U_{3A} of type 3A in $V^\natural$ (cf. [Mi3, LYY2, SY]). The 3A-algebra U_{3A} contains a unique extended $c = 4/5$ Virasoro sub-VOA $\mathcal{W} = \mathcal{W}(4/5)$ and the corresponding commutant subalgebra $\mathrm{Com}_{U_{3A}}(\mathcal{W})$ in U_{3A} (cf. [SY]) gives rise to certain $c = 6/7$ Virasoro vectors in $VF^\natural$, which we call *derived* Virasoro vectors (cf. Definition 4.6). Note that $\mathcal{W} \subset U_{3A} \subset V^\natural$ implies $VF^\natural = \mathrm{Com}_{V^\natural}(\mathcal{W}) \supset \mathrm{Com}_{U_{3A}}(\mathcal{W})$. It is interesting that a natural construction in [DLMN] suggests $c = 6/7$ Virasoro vectors in the case of E_6; cf. the discussion below. Motivated by the above observation, we first study subalgebras U_{3A} of any vertex operator algebra V and we show that one can canonically associate involutive automorphisms to derived $c = 6/7$ Virasoro vectors in $\mathrm{Com}_V(\mathcal{W})$, which we call *Miyamoto involutions* (see Lemma 2.4 and Eq. (2.4)). We will show that the collection of Miyamoto involutions associated to derived $c = 6/7$ Virasoro vectors generates a 3-transposition group.

We say that a VOA W over $\mathbb{R}$ is *compact* if W has a positive definite invariant bilinear form. A real sub-VOA W is said to be a *compact real form* of a VOA V over $\mathbb{C}$ if W is compact and $V \simeq \mathbb{C} \otimes_\mathbb{R} W$.

Main Theorem 2 (Theorem 4.10)**.** *Let* $V = \bigoplus_{\geq 0} V_n$ *be a VOA. Suppose that* $\dim V_0 = 1$, $V_1 = 0$ *and* V *has a compact real form* $V_\mathbb{R}$ *and every simple* $c = 1/2$ *Virasoro vector of* V

is in $V_{\mathbb{R}}$. Then the Miyamoto involutions associated to derived $c = 6/7$ Virasoro vectors in the commutant subalgebra $\mathrm{Com}_V(\mathcal{W}(4/5))$ *satisfy a 3-transposition property.*

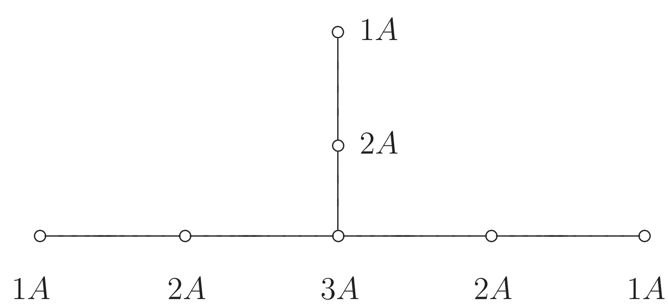
The Moonshine vertex operator algebra satisfies the assumption of the theorem above and we recover in a general fashion the 3-transposition property of the largest Fischer group via the commutant subalgebra $V\!F^\natural = \mathrm{Com}_{V^\natural}(\mathcal{W}(4/5))$ (see Corollary 5.12). Indicated by the fact that 3-transpositions are induced by $c = 6/7$ derived Virasoro vectors, we will also prove the following one-to-one correspondence.

Main Theorem 3 (Theorem 5.10)**.** *There exists a one-to-one correspondence between 3-transpositions of the Fischer group and derived $c = 6/7$ Virasoro vectors in $V\!F^\natural$ via Miyamoto involutions.*

This theorem provides a link for studying the 3-transpositions of Fi_{24} by using the VOA $V\!F^\natural$ and it is possible to relate McKay's E_6-observation of Fi_{24} to the theory of vertex operator algebra as discussed below.

McKay's observation. We are interested in the E_6 case of the observation of McKay which relates the Monster group and some sporadic groups involved in the Monster to the affine Dynkin diagrams of types E_6, E_7 and E_8 [Mc].

McKay's observation in the case of the largest Fischer group says that the orders of the products of any two 3-transpositions of Fi_{24} belongs to one of the conjugacy classes 1A, 2A or 3A of Fi_{24} such that these conjugacy classes coincide with the numerical labels of the nodes in an affine E_6 Dynkin diagram and there is a correspondence as follows:



This correspondence is not one-to-one but only up to diagram automorphisms.

For the understanding of the E_8-case [LM, LYY1, LYY2], the main foothold is the one-to-one correspondence between 2A-involutions of the Monster and simple $c = 1/2$ Virasoro vectors in $V^\natural$ by which one can translate McKay's E_8-observation into a purely vertex operator algebra theoretical problem. For the E_6 case, we have a nice correspondence as in Main Theorem 3 and we can also translate the E_6-observation into a problem of vertex operator algebras. Based on this correspondence and making use of the 3A-algebra

U_{3A}, which is generated by two $c = 1/2$ Virasoro vectors, we show that there is a natural connection between dihedral subgroups of Fi_{24} and certain sub-VOAs constructed by the nodes of the affine E_6 diagram, which gives some context of McKay's observation in terms of vertex operator algebras.

More precisely, let S be a simple root lattice with a simply laced root system $\Phi(S)$. We scale S such that the roots have squared length 2. Let $V_{\sqrt{2}S}$ be the lattice VOA associated to $\sqrt{2}S$. Here and further we use the standard notation for lattice VOAs as in [FLM]. In [DLMN] Dong et al. constructed a Virasoro vector of $V_{\sqrt{2}S}$ of the form

$$\tilde{\omega}_S := \frac{1}{2h(h+2)} \sum_{\alpha \in \Phi(S)} \alpha(-1)^2 \mathbb{1} + \frac{1}{h+2} \sum_{\alpha \in \Phi(S)} e^{\sqrt{2}\alpha}, \tag{1.1}$$

where h denotes the Coxeter number of S. Recall that the central charge of $\tilde{\omega}_S$ is $6/7$ if $S = E_6$ [DLMN]. By the expression, it is clear that $\tilde{\omega}_S$ is invariant under the Weyl group of $\Phi(S)$.

Our approach to McKay's observation is to find suitable pairs of derived $c = 6/7$ Virasoro vectors in $V\!F^\natural$ which inherits the E_6 structure in McKay's E_6-diagram. By using similar ideas as in [LYY1, LYY2], we construct a certain sub-VOA $U_{F(nX)}$ of the lattice VOA $V_{\sqrt{2}E_6}$ associated to each node nX of the affine E_6 diagram (cf. Section 3). Utilizing an embedding of the E_6 lattice into the E_8 lattice, we show that $U_{F(nX)}$ is contained in the VOA $V\!F^\natural$ purely by their VOA structures (cf. Theorem 5.16 and Appendix A). These sub-VOAs contain pairs of derived $c = 6/7$ Virasoro vectors such that the corresponding Miyamoto involutions generate a dihedral group of type nX in $\mathrm{Fi}_{24} \subset \mathrm{Aut}(V\!F^\natural)$. Then, using the identification of Fi_{24} as a subgroup of $\mathrm{Aut}(V\!F^\natural)$, we obtain another main result with the help of the Atlas [ATLAS].

Main Theorem 4 (Theorem 5.16)**.** *For any of the cases $nX = 1A$, $2A$ or $3A$, the VOA $U_{F(nX)}$ can be embedded into $V\!F^\natural$. Moreover, $\sigma_{\tilde{v}}\sigma_{\tilde{v}'}$ belongs to the conjugacy class nX of* $\mathrm{Fi}_{24} = \mathrm{Aut}(\mathfrak{X})$*, where $\tilde{v}$ and $\tilde{v}'$ are defined as in* (3.16)*.*

In this way, our embeddings of $U_{F(nX)}$ into $V\!F^\natural$ encode the E_6 structure into $V\!F^\natural$ which are compatible with the original McKay observations.

The organization of the paper. The organization of this article is as follows: In Section 2, we review basic properties about Virasoro VOAs and Virasoro vectors.

In Section 3, we recall the definition of commutant sub-VOAs and define certain commutant subalgebras associated to the root lattice of type E_6 using the method described in [LYY1, LYY2].

In Section 4, we study a vertex operator algebra U_{3A}, which we call the 3A-algebra for the Monster, and prove a 3-transposition property for Miyamoto involutions associated to derived $c = 6/7$ Virasoro vectors on commutant subalgebras of VOAs containing U_{3A}.

In Section 5, the commutant subalgebra $V\!F^\natural$ of $\mathcal{W}(4/5)$ in $V^\natural$ is studied. The sub-VOA $V\!F^\natural$ has a natural faithful action of the Fischer group Fi_{24}. We expect that the full automorphism group of $V\!F^\natural$ is Fi_{24} but we cannot give a proof of this prospect. Instead, we will show a partial result that the full automorphism group of the sub-VOA $\mathcal{X}$ generated by the weight two subspace of $V\!F^\natural$ is isomorphic to Fi_{24}. We also establish a one-to-one correspondence between 2C-involutions of Fi_{24} and derived $c = 6/7$ Virasoro vectors in $V\!F^\natural$.

Finally, we discuss the embeddings of the commutant subalgebras constructed in Section 3 into $V\!F^\natural$ in Section 5.3. We show that the $c = 6/7$ Virasoro vectors defined in Section 3.1 can be embedded into $V\!F^\natural$. Moreover, we verify that the product of the corresponding σ-involutions belongs to the conjugacy class associated to the node.

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Notation and Terminology. In this article, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of non-negative integers, integers, real and complex numbers, respectively. We denote the ring $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}_p$ with a positive integer p and often identify the integers $0, 1, \ldots, p-1$ with their images in $\mathbb{Z}_p$.

Every vertex operator algebra (VOA for short) is defined over the field $\mathbb{C}$ of complex numbers unless otherwise stated. A VOA V is called *of CFT-type* if it is non-negatively graded $V = \bigoplus_{n\geq 0} V_n$ with $V_0 = \mathbb{C}\mathbb{1}$. For a VOA structure $(V, Y(\cdot, z), \mathbb{1}, \omega)$ on V, the vector ω is called the *conformal vector* of V [1]. For simplicity, we often use (V, ω) to denote the structure $(V, Y(\cdot, z), \mathbb{1}, \omega)$. The vertex operator $Y(a, z)$ of $a \in V$ is expanded as $Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$.

[1] The conformal vector of V is often called the Virasoro element of V, e.g. [FLM]

An element $e \in V$ is referred to as a *Virasoro vector of central charge* $c_e \in \mathbb{C}$ if $e \in V_2$ and it satisfies $e_{(1)}e = 2e$ and $e_{(3)}e = (c_e/2)\cdot \mathbb{1}$. It is well-known that the associated modes $L^e(n) := e_{(n+1)}$, $n \in \mathbb{Z}$, generate a representation of the Virasoro algebra on V (cf. [Mi1]), i.e., they satisfy the commutator relation

$$[L^e(m), L^e(n)] = (m-n)L^e(m+n) + \delta_{m+n,0}\frac{m^3-m}{12}c_e.$$

Therefore, a Virasoro vector together with the vacuum vector generates a Virasoro VOA inside V. We will denote this subalgebra by $\mathrm{Vir}(e)$.

In this paper, we define a sub-VOA of V to be a pair (U, e) consisting of a subalgebra U containing the vacuum element $\mathbb{1}$ and a conformal vector e for U such that (U, e) inherits the grading of V, that is, $U = \bigoplus_{n\ge 0} U_n$ with $U_n = V_n \cap U$, but e may not be the conformal vector of V. In the case that e is also the conformal vector of V, we will call the sub-VOA (U, e) a *full* sub-VOA.

For a positive definite even lattice L, we will denote the lattice VOA associated to L by V_L (cf. [FLM]). We adopt the standard notation for V_L as in [FLM]. In particular, V_L^+ denotes the fixed point subalgebra of V_L under a lift of the (-1)-isometry on L. The letter Λ always denotes the Leech lattice, the unique even unimodular lattice of rank 24 without roots.

Given a group G of automorphisms of V, we denote by V^G the fixed point subalgebra of G in V. The subalgebra V^G is called the *G-orbifold* of V in the literature. For a V-module $(M, Y_M(\cdot, z))$ and $\sigma \in \mathrm{Aut}(V)$, we set ${}^{\sigma}Y_M(a, z) := Y_M(\sigma^{-1}a, z)$ for $a \in V$. Then the *σ-conjugated module* $\sigma \circ M$ of M is defined to be the module structure $(M, {}^{\sigma}Y_M(\cdot, z))$.

2 Virasoro vertex operator algebras and their extensions

For complex numbers c and h, we denote by $L(c, h)$ the irreducible highest weight representation of the Virasoro algebra with central charge c and highest weight h. It is shown in [FZ] that $L(c, 0)$ has a natural structure of a simple VOA.

2.1 Unitary Virasoro vertex operator algebras

Let

$$\begin{aligned} c_m &:= 1 - \frac{6}{(m+2)(m+3)}, \qquad m = 1, 2, \dots, \\ h_{r,s}^{(m)} &:= \frac{\{r(m+3) - s(m+2)\}^2 - 1}{4(m+2)(m+3)}, \quad 1 \le s \le r \le m+1. \end{aligned} \tag{2.1}$$

It is shown in [W] that $L(c_m, 0)$ is rational and $L(c_m, h^{(m)}_{r,s})$, $1 \le s \le r \le m+1$, provide all irreducible $L(c_m, 0)$-modules (see also [DMZ]). This is the so-called unitary series of the Virasoro VOAs. The fusion rules among $L(c_m, 0)$-modules are computed in [W] and given by

$$L(c_m, h^{(m)}_{r_1,s_1}) \times L(c_m, h^{(m)}_{r_2,s_2}) = \sum_{\substack{i\in I\\ j\in J}} L(c_m, h^{(m)}_{|r_1-r_2|+2i-1, |s_1-s_2|+2j-1}), \tag{2.2}$$

where

$$I = \{1\,, 2,\, \ldots,\, \min\{r_1,\, r_2,\, m+2-r_1,\, m+2-r_2\}\},$$

$$J = \{1,\, 2,\, \ldots,\, \min\{s_1,\, s_2,\, m+3-s_1,\, m+3-s_2\}\}.$$

Definition 2.1. A Virasoro vector e with central charge c is called *simple* if $\mathrm{Vir}(e) \simeq L(c, 0)$. A simple $c = 1/2$ Virasoro vector is called an *Ising vector.*

The fusion rules among $L(c_m, 0)$-modules have a canonical $\mathbb{Z}_2$-symmetry and this symmetry gives rise to an involutive vertex operator algebra automorphism which is known as Miyamoto involution.

Theorem 2.2 ([Mi1])**.** *Let V be a VOA and let $e \in V$ be a simple Virasoro vector with central charge c_m. Denote by $V_e[h^{(m)}_{r,s}]$ the sum of irreducible* $\mathrm{Vir}(e) = L(c_m, 0)$*-submodules of V isomorphic to $L(c_m, h^{(m)}_{r,s})$, $1 \le s \le r \le m+1$. Then the linear map*

$$\tau_e := \begin{cases} (-1)^{r+1} & \text{on } V_e[h^{(m)}_{r,s}] \quad \text{if } m \text{ is even,} \\ (-1)^{s+1} & \text{on } V_e[h^{(m)}_{r,s}] \quad \text{if } m \text{ is odd,} \end{cases}$$

defines an automorphism of V called the τ-involution associated to e.

Next we introduce a notion of σ-type Virasoro vectors.

Definition 2.3. For $m = 1, 2, \ldots$, let

$$B^{(m)} := \begin{cases} \left\{ h^{(m)}_{1,s} \mid 1 \le s \le m+2 \right\} & \text{if } m \text{ is even,} \\ \left\{ h^{(m)}_{r,1} \mid 1 \le r \le m+1 \right\} & \text{if } m \text{ is odd.} \end{cases}$$

A simple Virasoro vector $e \in V$ with central charge c_m is said to be of σ*-type* on V if $V_e[h] = 0$ for all $h \notin B^{(m)}$.

By Eq. (2.2), the fusion rules among irreducible modules $L(c_m, h)$, $h \in B^{(m)}$, are relatively simple. Moreover, $B^{(m)}$ possesses a natural $\mathbb{Z}_2$-symmetry as follows.

Lemma 2.4. *Let $e \in V$ be a simple $c = c_m$ Virasoro vector of σ-type. Then one has the isotypical decomposition*

$$V = \bigoplus_{h \in B^{(m)}} V_e[h]$$

and the linear map σ_e given by

$$\sigma_e := \begin{cases} (-1)^{s+1} & \text{on } \ V_e[h_{1,s}^{(m)}] \quad \text{if } m \text{ is even,} \\ (-1)^{r+1} & \text{on } \ V_e[h_{r,1}^{(m)}] \quad \text{if } m \text{ is odd.} \end{cases} \tag{2.3}$$

is an automorphism of V.

We will call the map σ_e above the *Miyamoto involution (of σ-type)* associated to a simple $c = c_m$ Virasoro vector e of σ-type.

Remark 2.5. (1) By Eq. (2.2), $B^{(m)}$ is closed under the fusion product, i.e., if $h,\, h' \in B^{(m)}$ then $L(c_m, h) \times L(c_m, h')$ is a sum of irreducible modules with highest weights in $B^{(m)}$. Therefore, the subspace $W = \bigoplus_{h \in B^{(m)}} V_e[h]$ forms a sub-VOA of V. Note that e is of σ-type on W and one can always define σ_e as an automorphism of W.
(2) By definition, it is clear that τ_e acts trivially on $V_e[h]$ for all $h \in B^{(m)}$. However, the fixed point sub-VOA of τ_e on V is usually bigger than $W = \bigoplus_{h \in B^{(m)}} V_e[h]$.

In this article, we will mainly consider the case $c_4 = 6/7$. In this case, $B^{(4)} = \{0,\, 5,\, 1/7,\, 5/7,\, 12/7,\, 22/7\}$ and a simple $c = 6/7$ Virasoro vector $e \in V$ is of σ-type on V if $V_e[h] = 0$ for $h \neq 0,\, 5,\, 1/7,\, 5/7,\, 12/7,\, 22/7$. The corresponding σ-involution is given by

$$\sigma_e := \begin{cases} \ \ 1 & \text{on } \ V_e[0] \oplus V_e[5/7] \oplus V_e[22/7], \\ -1 & \text{on } \ V_e[5] \oplus V_e[12/7] \oplus V_e[1/7]. \end{cases} \tag{2.4}$$

We also need the following result:

Lemma 2.6. *Let V be a VOA with grading $V = \bigoplus_{n \geq 0} V_n$, $V_0 = \mathbb{C}\mathbb{1}$ and $V_1 = 0$, and let $u \in V$ be a Virasoro vector such that $\mathrm{Vir}(u) \simeq L(c_m, 0)$. Then the zero mode $o(u) = u_{(1)}$ acts on the Griess algebra of V semisimply with possible eigenvalues 2 and $h_{r,s}^{(m)}$, $1 \leq s \leq r \leq m+1$. Moreover, if $h_{r,s}^{(m)} \neq 2$ for $1 \leq s \leq r \leq m+1$ then the eigenspace for the eigenvalue 2 is one-dimensional, namely, it is spanned by the Virasoro vector u.*

Proof: See Lemma 2.6 of [HLY]. ∎

2.2 Extended Virasoro vertex operator algebras

Among $L(c_m, 0)$-modules, only $L(c_m, 0)$ and $L(c_m, h^{(m)}_{m+1,1})$ are simple currents, and it is shown in [LLY] that $L(c_m, 0) \oplus L(c_m, h^{(m)}_{m+1,1})$ forms a simple current extension of $L(c_m, 0)$. Note that $h^{(m)}_{m+1,1} (= h^{(m)}_{1,m+2}) = m(m + 1)/4$ is an integer if $m \equiv 0, 3 \pmod 4$ and a half-integer if $m \equiv 1, 2 \pmod 4$.

Theorem 2.7 ([LLY]). *The $\mathbb{Z}_2$-graded simple current extension*

$$\mathcal{W}(c_m) := L(c_m, 0) \oplus L(c_m, h^{(m)}_{m+1,1})$$

has a unique simple rational vertex operator algebra structure extending $L(c_m, 0)$ if $m \equiv 0, 3 \pmod 4$, and a unique simple rational vertex operator superalgebra structure extending $L(c_m, 0)$ if $m \equiv 1, 2 \pmod 4$.

By this theorem, we introduce the following notion.

Definition 2.8. Let $m \equiv 0$ or $3 \pmod 4$. A simple $c = c_m$ Virasoro vector u of a VOA V is called *extendable* if there exists a non-zero highest weight vector $w \in V$ of weight $h^{(m)}_{m+1,1} = m(m+1)/4$ with respect to $\mathrm{Vir}(u)$ such that the subalgebra generated by u and w is isomorphic to the extended Virasoro VOA $\mathcal{W}(c_m)$. Note that w is a $\mathrm{Vir}(u)$-primary vector, i.e., $L^u(m)w = 0$ for all $m > 0$, where $L^u(m) = u_{(m+1)}$. We will call such a w an $h^{(m)}_{m+1,1}$*-primary vector* associated to u.

Lemma 2.9. *Let $m \equiv 0, 3 \pmod 4$ and $u \in V$ be a simple extendable $c = c_m$ Virasoro vector. Then an $h^{(m)}_{m+1,1}$-primary vector associated to u is unique up to scalar multiple.*

Proof: Let w, w' be $h^{(m)}_{m+1,1}$-primary vectors associated to u and $\mathcal{W}$ the subalgebra generated by u and w. Then the $\mathcal{W}$-submodule $\mathcal{W}'$ generated by w' is isomorphic to the adjoint module $\mathcal{W}$. Since we have assumed that $V_0 = \mathbb{C}\mathbb{1}$, we see that $\mathcal{W} = \mathcal{W}'$ and the assertion follows. ∎

Next we discuss the irreducible modules for $\mathcal{W}(c_m)$ when $m \equiv 0$ or $3 \pmod 4$.

Lemma 2.10 ([LLY]). *Let $L(c_m, h^{(m)}_{m+1,1}) \times L(c_m, h^{(m)}_{r,s}) = L(c_m, \tilde{h}^{(m)}_{r,s})$, where*

$$\tilde{h}^{(m)}_{r,s} = \begin{cases} h^{(m)}_{m+2-r,s} & \text{when } m \equiv 0 \pmod 4, \\ h^{(m)}_{r,m+3-s} & \text{when } m \equiv 3 \pmod 4, \end{cases}$$

by Eq. (2.2). Then $\Delta = h^{(m)}_{r,s} - \tilde{h}^{(m)}_{r,s}$ is in either $\mathbb{Z}$ or $\frac{1}{2} + \mathbb{Z}$. More precisely, one has:
(1) When $m \equiv 0 \pmod 4$, $\Delta \in \mathbb{Z}$ if r is odd, and $\Delta \in \frac{1}{2} + \mathbb{Z}$ if r is even.
(2) When $m \equiv 3 \pmod 4$, $\Delta \in \mathbb{Z}$ if s is odd, and $\Delta \in \frac{1}{2} + \mathbb{Z}$ if s is even.
(3) $\Delta = 0$, i.e., $h^{(m)}_{r,s} = \tilde{h}^{(m)}_{r,s}$ when the triple (m, r, s) satisfies $r = (m + 2)/2$ if $m \equiv 0 \pmod 4$ or $s = (m + 3)/2$ if $m \equiv 3 \pmod 4$.

Theorem 2.11 ([LLY]). *Suppose $m \equiv 0$ or $3 \pmod 4$. Let $M = L(c_m, h^{(m)}_{r,s})$ be an irreducible $L(c_m, 0)$-module and $\tilde{M} = L(c_m, \tilde{h}^{(m)}_{r,s}) := L(c_m, h^{(m)}_{m+1,1}) \times M$.*
(1) If $h^{(m)}_{r,s} - \tilde{h}^{(m)}_{r,s} \in \mathbb{Z} \setminus \{0\}$, then $M \oplus \tilde{M}$ affords a unique structure of an irreducible (untwisted) $\mathcal{W}(c_m)$-module extending M.
(2) If $h^{(m)}_{r,s} - \tilde{h}^{(m)}_{r,s} \in \frac{1}{2} + \mathbb{Z}$, then $M \oplus \tilde{M}$ affords a unique structure of an irreducible $\mathbb{Z}_2$-twisted $\mathcal{W}(c_m)$-module extending M.
(3) If $h^{(m)}_{r,s} = \tilde{h}^{(m)}_{r,s}$, then $M \oplus \tilde{M}$ is a direct sum of two inequivalent irreducible (untwisted) $\mathcal{W}(c_m)$-modules. In this case, there exists two inequivalent structures of an irreducible (untwisted) $\mathcal{W}(c_m)$-module on M and these structures are $\mathbb{Z}_2$-conjugates of each other. We denote them by $M^{\pm}$.

Note that Theorem 2.11 together with Lemma 2.10 provides a classification of irreducible $\mathcal{W}(c_m)$-modules.

Remark 2.12. If u is a simple extendable $c = c_m$ Virasoro vector of V, then it follows from Lemma 2.10 and Theorem 2.11 that the automorphism τ_u defined in Theorem 2.2 is trivial on V since V is an untwisted module over the extended subalgebra $\mathcal{W}(c_m)$ of $\mathrm{Vir}(u)$.

In this paper, we will frequently consider simple extendable Virasoro vectors with central charges $c_3 = 4/5$ and $c_4 = 6/7$. The key feature is that the extended Virasoro VOAs $\mathcal{W}(4/5)$ and $\mathcal{W}(6/7)$ have some natural $\mathbb{Z}_3$-symmetries among their irreducible modules.

The extended Virasoro VOA $\mathcal{W}(4/5)$ is rational and has six inequivalent irreducible modules (cf. [KMY]). They are of the following forms as $L(4/5, 0)$-modules:

$$L(4/5, 0) \oplus L(4/5, 3), \quad L(4/5, 2/5) \oplus L(4/5, 7/5), \quad L(4/5, 2/3)^{\pm}, \quad L(4/5, 1/15)^{\pm},$$

where the ambiguity on choosing signs $\pm$ is solved by fusion rules (cf. [Mi2]). The extended Virasoro VOA $\mathcal{W}(6/7)$ is rational and has nine inequivalent irreducible modules (cf. [LLY, LY]). They are of the following forms as $L(6/7, 0)$-modules:

$$\begin{array}{lll} L(6/7, 0) \oplus L(6/7, 5), & L(6/7, 1/7) \oplus L(6/7, 22/7), & L(6/7, 5/7) \oplus L(6/7, 12/7), \\ L(6/7, 4/3)^{\pm}, & L(6/7, 1/21)^{\pm}, & L(6/7, 10/21)^{\pm}, \end{array}$$

where the ambiguity on choosing signs $\pm$ is again solved by fusion rules (cf. loc. cit.). The fusion rules among irreducible $\mathcal{W}(4/5)$-modules and $\mathcal{W}(6/7)$-modules are computed in [Mi2, LLY, LY] and they have some natural $\mathbb{Z}_3$-symmetries. We can extend these symmetries to automorphisms of VOAs containing these extended Virasoro VOAs as follows.

Theorem 2.13 ([Mi2, LLY, LY]). *Let V be a VOA and let U be a sub-VOA of V.*
(1) Suppose $U \simeq \mathcal{W}(4/5)$. Define a linear automorphism ξ_U of V to act on each irreducible $\mathcal{W}(4/5)$-submodule M by

$$\begin{cases} 1 & \text{if } M \simeq L(4/5,0) \oplus L(4/5,3) \text{ or } L(4/5,2/5) \oplus L(4/5,7/5), \\ e^{\pm 2\pi\sqrt{-1}/3} & \text{if } M \simeq L(4/5,2/3)^{\pm} \text{ or } L(4/5,1/15)^{\pm} \end{cases}$$

as $L(4/5,0)$-modules. Then ξ_U defines an element in $\mathrm{Aut}(V)$ satisfying ${\xi_U}^3 = 1$.
(2) Suppose $U \simeq \mathcal{W}(6/7)$. Define a linear automorphism ξ_U of V to act on each irreducible $\mathcal{W}(6/7)$-submodule M by

$$\begin{cases} 1 & \text{if } M \simeq L(6/7,0) \oplus L(6/7,5),\ L(6/7,1/7) \oplus L(6/7,22/7) \text{ or } L(6/7,5/7) \oplus L(6/7,12/7), \\ e^{\pm 2\pi\sqrt{-1}/3} & \text{if } M \simeq L(6/7,4/3)^{\pm},\ L(6/7,1/21)^{\pm} \text{ or } L(6/7,10/21)^{\pm} \end{cases}$$

as $L(6/7,0)$-modules. Then ξ_U defines an element in $\mathrm{Aut}(V)$ satisfying ${\xi_U}^3 = 1$.

Remark 2.14. If a simple $c = 6/7$ Virasoro vector $x \in V$ is extendable, then x is of σ-type if and only if the automorphism ξ_U defined in (2) of Theorem 2.13 is trivial on V, where U is the subalgebra isomorphic to $\mathcal{W}(6/7)$ generated by x and its 5-primary vector.

3 Commutant subalgebras associated to root lattices

In this section, we will construct sub-VOAs of the lattice VOA $V_{\sqrt{2}E_6}$ which will correspond to dihedral subgroups of the largest Fischer group. Our construction is similar to the construction in [LYY1] and [HLY] in the case of the root lattices E_8 and E_7.

3.1 The algebras $U_{F(nX)}$ and $V_{F(nX)}$

Commutant subalgebras. Let (V,ω) be a VOA and (U,e) be a sub-VOA. Then the commutant subalgebra of U is defined by

$$\mathrm{Com}_V(U) := \{a \in V \mid a_{(n)}U = 0 \text{ for all } n \geq 0\}. \tag{3.1}$$

It is known (cf. [FZ]) that

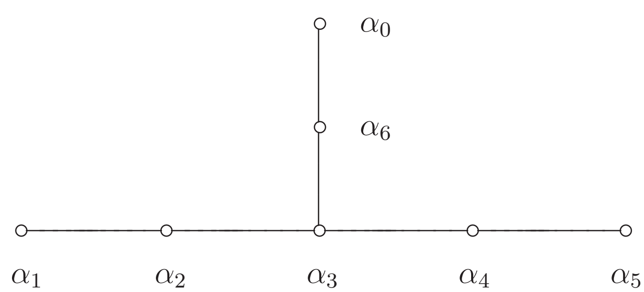
$$\mathrm{Com}_V(U) = \ker_V e_{(0)} \tag{3.2}$$

and in particular $\mathrm{Com}_V(U) = \mathrm{Com}_V(\mathrm{Vir}(e))$. Therefore, the commutant subalgebra of U is determined only by the conformal vector e of U. It is also shown in Theorem 5.1 of [FZ] that $\omega - e$ is also a Virasoro vector if $\omega_{(2)}e = 0$. Provided $V_1 = 0$, we always have $\omega_{(2)}e = 0$. In that case, we have two mutually commuting subalgebras $\mathrm{Com}_V(\mathrm{Vir}(e)) = \ker_V e_{(0)}$ and $\mathrm{Com}_V(\mathrm{Vir}(\omega - e)) = \ker_V(\omega - e)_{(0)}$ and the tensor product $\mathrm{Com}_V(\mathrm{Vir}(\omega - e)) \otimes \mathrm{Com}_V(\mathrm{Vir}(e))$ forms an extension of $\mathrm{Vir}(e) \otimes \mathrm{Vir}(\omega - e)$. More generally, we say a sum $\omega = e^1 + \cdots + e^n$ is a *Virasoro frame* if all e^i are Virasoro vectors and $[Y(e^i, z_1), Y(e^j, z_2)] = 0$ for $i \neq j$.

The algebras $U_{F(nX)}$. Let $\alpha_1, \ldots, \alpha_6$ be a system of simple roots for E_6. We let α_0 be the root such that $-\alpha_0 = \sum_{i=1}^{6} m_i\alpha_i$ is the highest root for the chosen simple roots. Note that all m_i are positive integers. We also set $m_0 = 1$. For any $i = 0, \ldots, 6$, we consider the sublattice L_i of E_6 generated by the roots α_j, $0 \leq j \leq 6$, $j \neq i$. One observes that L_i is also of rank 6 and the quotient group E_6/L_i is cyclic of order m_i with generator $\alpha_i + L_i$. Thus one has

$$E_6 = L_i \sqcup (\alpha_i + L_i) \sqcup (2\alpha_i + L_i) \sqcup \cdots \sqcup ((m_i - 1)\alpha_i + L_i). \tag{3.3}$$

We denote by $R_1, \ldots, R_\ell$ the indecomposable components of the lattice L_i which are root lattices of type A_n, D_n or E_6. Hence $L_i = R_1 \oplus \cdots \oplus R_\ell$ where the direct sum of lattices denotes the orthogonal sum. In fact, the Dynkin diagram of L_i is obtained from the affine Dynkin diagram of E_6 by removing the node α_i and the adjacent edges. We recall here that the affine Dynkin diagram of E_6 is the graph with vertex set $\{\alpha_0, \ldots, \alpha_6\}$ and two nodes α_i and α_j, $0 \leq i, j \leq 6$, are joined by an edge if $\langle \alpha_i, \alpha_j \rangle = -1$. The diagram has the following form:



(3.4)

The decomposition (3.3) of the lattice E_6 leads to the decomposition

$$V_{\sqrt{2}E_6} = \bigoplus_{r=0}^{m_i - 1} V_{\sqrt{2}(r\alpha_i + L_i)}$$

of the lattice VOA $V_{\sqrt{2}E_6}$. We define a linear map $\rho_i : V_{\sqrt{2}E_6} \to V_{\sqrt{2}E_6}$ by

$$\rho_i(u) = \zeta_{m_i}^r u \quad \text{for } u \in V_{\sqrt{2}(r\alpha_i + L_i)}, \qquad \text{where } \zeta_{m_i} = e^{2\pi\sqrt{-1}/m_i}. \tag{3.5}$$

Then ρ_i is an element of $\mathrm{Aut}(V_{\sqrt{2}E_6})$ of order m_i and the fixed point sub-VOA $V_{\sqrt{2}E_6}^{\langle \rho_i \rangle}$ is exactly $V_{\sqrt{2}L_i}$.

For a root lattice S, we denote by $\Phi(S)$ its root system. Then, by [DLMN], the conformal vector ω_S of $V_{\sqrt{2}S}$ is given by

$$\omega_S = \frac{1}{4h} \sum_{\alpha \in \Phi(S)} \alpha(-1)^2 \mathbb{1},$$

where h is the Coxeter number of S. Now define

$$\tilde{\omega}_S := \frac{2}{h+2}\,\omega_S + \frac{1}{h+2} \sum_{\alpha \in \Phi(S)} e^{\sqrt{2}\alpha}. \tag{3.6}$$

It is shown in [DLMN] that $\tilde{\omega}_S$ is a Virasoro vector of central charge $2n/(n+3)$ if S is of type A_n, 1 if S is of type D_n, and $6/7$, $7/10$ and $1/2$ if S is of type E_6, E_7, E_8, respectively. From the irreducible decomposition $L_i = R_1 \oplus \cdots \oplus R_\ell \subset E_6$, we have sublattices R_s of E_6 and obtain a factorization

$$V_{\sqrt{2}L_i} = V_{\sqrt{2}R_1} \otimes \cdots \otimes V_{\sqrt{2}R_\ell} \subset V_{\sqrt{2}E_6}. \tag{3.7}$$

Associated to the root subsystems $\Phi(R_s)$ of $\Phi(E_6)$, we also have Virasoro vectors

$$\omega^s := \tilde{\omega}_{R_s} = \frac{2}{h_s+2}\,\omega_{R_s} + \frac{1}{h_s+2} \sum_{\alpha \in \Phi(R_s)} e^{\sqrt{2}\alpha} \in V_{\sqrt{2}R_s} \subset V_{\sqrt{2}E_6}, \quad 1 \le s \le \ell, \tag{3.8}$$

where ω_{R_s} is the conformal vector of $V_{\sqrt{2}R_s}$ and h_s is the Coxeter number of R_s. It follows from the definition that $\omega^s, 1 \le s \le \ell$, are mutually orthogonal simple Virasoro vectors in $V_{\sqrt{2}E_6}$. Consider

$$X^r := \sum_{\substack{\beta \in r\alpha_i + L_i \\ \langle \beta, \beta \rangle = 2}} e^{\sqrt{2}\beta}, \qquad 1 \le r \le m_i - 1,$$

in the weight two subspace of $V_{\sqrt{2}E_6}$. It is shown in Proposition 2.2 of [LYY1] that the vectors X^r are highest weight vectors for $\mathrm{Vir}(\omega^1) \otimes \cdots \otimes \mathrm{Vir}(\omega^\ell)$ with total weight 2.

Since all ω^s, $1 \le s \le \ell$, are contained in the fixed point sub-VOA $V^+_{\sqrt{2}E_6}$, which has a trivial weight one subspace, the vector $\omega_{E_6} - (\omega^1 + \cdots + \omega^\ell)$ is a Virasoro vector of $V_{\sqrt{2}E_6}$ as discussed at the beginning of the section. We are interested in the following commutant subalgebras:

Definition 3.1. For $i = 0, \ldots, 6$, let $L_i < E_6$ be defined as in (3.3) and $R_1, \ldots, R_\ell$ the indecomposable components of L_i. Let $\omega^s = \tilde{\omega}_{R_s}$ be Virasoro vectors defined as in (3.8) for $1 \le s \le \ell$. The algebra $U(i)$ is the vertex operator algebra

$$\begin{aligned} U(i) &= \mathrm{Com}_{V_{\sqrt{2}E_6}}(\mathrm{Vir}(\omega_{E_6} - (\omega^1 + \cdots + \omega^\ell))) \\ &= \ker_{V_{\sqrt{2}E_6}} \left(\omega_{E_6} - (\omega^1 + \cdots + \omega^\ell)\right)_{(0)}. \end{aligned}$$

It is clear from the construction that $U(i)$ has a Virasoro frame $\omega^1 + \cdots + \omega^\ell$. We will consider an embedding of $U(i)$ into a larger VOA and then describe the commutant algebra $U(i)$ using the larger VOA.

It is clear that $U(i)$ forms an extension of $\mathrm{Vir}(\omega^1) \otimes \cdots \otimes \mathrm{Vir}(\omega^\ell)$ and contains highest weight vectors X^r, $1 \le r < m_i$. We will see in Section 5 that we can embed $U(i)$ into the Moonshine VOA and therefore $U(i)$ has a trivial weight one subspace. Consequently, the weight two subspace of $U(i)$ carries a structure of a commutative non-associative algebra called the Griess algebra of $U(i)$, even though $V_{\sqrt{2}E_6}$ has a non-trivial weight one subspace. In Section 3.2, we will explicitly describe the Griess algebra of $U(i)$. Namely, we will show that the Griess algebra of $U(i)$ is given by

$$\mathcal{G}(i) := \mathrm{span}_{\mathbb{C}}\{\ \omega^s,\ X^r \mid 1 \le s \le \ell,\ \ 1 \le r \le m_i - 1\},$$

which is of dimension $\ell + m_i - 1$.

Recall $\rho_i \in \mathrm{Aut}(V_{\sqrt{2}E_6})$ defined as in (3.5). By definition, it is clear that $\tilde{\omega}_{E_6}$ and $\rho_i \tilde{\omega}_{E_6}$ are linear combinations of ω^s and X^r, and hence are contained in $\mathcal{G}(i) \subset U(i)$. We will also discuss the structure of the subalgebra generated by $\tilde{\omega}_{E_6}$ and $\rho_i \tilde{\omega}_{E_6}$ and compare it with $\mathcal{G}(i)$ in Section 3.2.

The algebras $V_{F(nX)}$. We also consider another class of commutant algebras inside the VOA $V_{\sqrt{2}E_8}$. These commutant algebras will be used in Section 5.3 to show that $U(i)$ defined in Definition 3.1 can be embedded into the Fischer group VOA $V\!F^\natural$.

We fix an embedding of E_6 into E_8. Let

$$Q := \mathrm{Ann}_{E_8}(E_6) = \{\alpha \in E_8 \mid \langle \alpha, E_6 \rangle = 0\}. \tag{3.9}$$

Then $Q \simeq A_2$ and $Q \oplus E_6$ forms a full rank sublattice of E_8. Note that such an embedding is unique up to an automorphism of E_8.

Recall that L_i is the sublattice of E_6 generated by roots α_j, $j \ne i$. Then we have an embedding of $\tilde{L}_i := Q \oplus L_i$ into E_8. Since L_i is a full rank sublattice of E_6, $\tilde{L}_i$ is also a full rank sublattice of E_8. Thus $E_8/\tilde{L}_i$ is a finite abelian group whose order is $3m_i$. We fix the corresponding embedding $V_{\sqrt{2}\tilde{L}_i} \subset V_{\sqrt{2}E_8}$.

We have the decomposition $\tilde{L}_i = Q \oplus R_1 \oplus \cdots \oplus R_\ell$ into a sum of irreducible root lattices which gives rise to a factorization

$$V_{\sqrt{2}\tilde{L}_i} = V_{\sqrt{2}Q} \otimes V_{\sqrt{2}L_i} = V_{\sqrt{2}Q} \otimes V_{\sqrt{2}R_1} \otimes \cdots \otimes V_{\sqrt{2}R_\ell} \subset V_{\sqrt{2}E_8}.$$

Let ω_{E_8} be the conformal vector of $V_{\sqrt{2}E_8}$ and let $\tilde{\omega}_Q \in V_{\sqrt{2}Q}$ and $\omega^s \in V_{\sqrt{2}R_s}$ be the Virasoro vectors defined as in (3.6) and (3.8), respectively. By the same argument as for

$U(i)$, one sees $\omega_{E_8} - (\tilde{\omega}_Q + \omega^1 + \cdots + \omega^\ell)$ is a Virasoro vector of $V_{\sqrt{2}E_8}$, and we can define a commutant subalgebra:

Definition 3.2. The algebra $V(i)$ is the commutant subalgebra

$$V(i) := \mathrm{Com}_{V_{\sqrt{2}E_8}}(\mathrm{Vir}(\omega_{E_8} - (\tilde{\omega}_Q + \omega^1 + \cdots + \omega^\ell))).$$

Remark 3.3. As explained in [HLY], $\mathrm{Com}_{V(i)}(\mathrm{Vir}(\tilde{\omega}_Q))$ coincides with $U(i)$.

We finally set

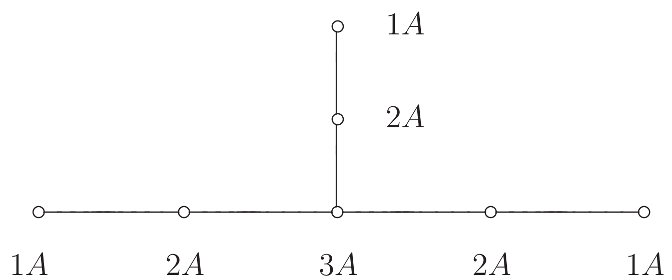
$$\mathcal{F}(i) := \{g \in \mathrm{Aut}(V_{\sqrt{2}E_8}) \mid g = \mathrm{id} \ \text{ on } \ V_{\sqrt{2}\tilde{L}_i}\}. \tag{3.10}$$

Then $\mathcal{F}(i)$ is canonically isomorphic to the group of characters of $E_8/\tilde{L}_i$. The subalgebra $V(i)$ of $V_{\sqrt{2}E_8}$ is invariant under the action of $\mathcal{F}(i)$ since all $\tilde{\omega}_Q$, ω^1, $\ldots$, ω^ℓ and the conformal vector ω_{E_8} of $V_{\sqrt{2}E_8}$ are clearly fixed by $\mathcal{F}(i)$. Note that the special Ising vector

$$\hat{e} := \tilde{\omega}_{E_8} = \frac{1}{16}\omega_{E_8} + \frac{1}{32}\sum_{\alpha\in\Phi(E_8)} e^{\sqrt{2}\alpha} \in V_{\sqrt{2}E_8} \tag{3.11}$$

is contained in $V(i)$ (cf. [LYY1, LYY2]) and thus $\{g\hat{e} \in V_{\sqrt{2}E_8} \mid g \in \mathcal{F}(i)\} \subset V(i)$.

McKay's E_6-correspondence. We like to explain McKay's correspondence between the conjugacy classes 1A, 2A and 3A of the Fischer group Fi_{24} which are the products of 2C-involutions of the Fischer group and the numerical labels m_i of the affine E_6 Dynkin diagram as given by the following figure:



Note that the correspondence is not one-to-one but only up to the diagram automorphism.

Because of this correspondence, we change our notation slightly and denote L_i by L_{nX}, ρ_i by ρ_{nX}, $\tilde{L}_i$ by $\tilde{L}_{nX}$, $\mathcal{F}(i)$ by $\mathcal{F}_{nX}$, $V(i)$ by $V_{F(nX)}$, $U(i)$ by $U_{F(nX)}$ and $\mathcal{G}(i)$ by $\mathcal{G}_{F(nX)}$, where $nX \in \{1A, 2A, 3A\}$ is the label of the corresponding node in (3.1). Explicitly, we have:

$$L_{1A} \simeq E_6, \qquad L_{2A} \simeq A_1 \oplus A_5, \qquad L_{3A} \simeq A_2 \oplus A_2 \oplus A_2. \tag{3.12}$$

We also have that

$$\tilde{\omega}_Q = \frac{1}{15}\left(\beta_0(-1)^2 + \beta_1(-1)^2 + \beta_2(-1)^2\right)\mathbb{1} + \frac{1}{5}\sum_{\alpha\in\Phi(Q)} e^{\sqrt{2}\alpha} \in V_{\sqrt{2}Q}$$

is a simple $c = 4/5$ Virasoro vector in $V_{\sqrt{2}Q}$, where $\{\beta_1, \beta_2\}$ is a set of simple roots for $Q \simeq A_2$ and $\beta_0 = -(\beta_1 + \beta_2)$.

3.2 Structures of $V_{F(nX)}$ and $U_{F(nX)}$.

We determine the structures of $V_{F(nX)}$ and $U_{F(nX)}$.

1A case. In this case, we have $\tilde{L}_{1A} \simeq A_2 \oplus E_6$ and $V_{F(1A)} \simeq U_{3A}$ by [LYY2]. It is clear that the following holds (see [LY] and Section 4 for details):

Lemma 3.4. $U_{F(1A)} \simeq \mathcal{W}(6/7) = L(6/7, 0) \oplus L(6/7, 5)$.

Therefore the weight two subspace of $U_{F(1A)}$ is one-dimensional.

2A case. In this case $\tilde{L}_{2A} \simeq A_2 \oplus A_1 \oplus A_5$ and $E_8/\tilde{L}_{3A} \simeq \mathbb{Z}_6$. By construction, the commutant subalgebra $V_{F(2A)}$ is the same as the monstrous 6A-algebra U_{6A} discussed in [LYY2] (see also the Appendix of [HLY]). Thus, the following result follows from [LYY2]:

Lemma 3.5. *There is the decomposition*

$$\begin{aligned} U_{F(2A)} \simeq \quad & \mathcal{W}(6/7) \otimes L(25/28, 0) \;\oplus\; \left(L(6/7, 5/7) \oplus L(6/7, 12/7)\right) \otimes L(25/28, 9/7) \\ & \oplus \left(L(6/7, 1/7) \oplus L(6/7, 22/7)\right) \otimes L(25/28, 34/7) \end{aligned}$$

as a $\mathcal{W}(6/7) \otimes L(25/28, 0)$*-module.*

It follows from the decomposition in the Lemma that the weight two subspace of $U_{F(2A)}$ is 3-dimensional and coincides with $\mathcal{G}_{F(2A)}$.

3A case. In this case, $\tilde{L}_{3A} \simeq A_2 \oplus A_2 \oplus A_2 \oplus A_2$ and $E_8/\tilde{L}_{3A} \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_3$. In fact, the coset structure $E_8/\tilde{L}_{3A}$ can be identified with the ternary *tetra code* $\mathcal{C}_4$, whose generator matrix is given by

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}.$$

The sub-VOA $V_{F(3A)}$ is indeed the ternary code VOA defined in [KMY]:

$$M_{\mathcal{C}_4} = \bigoplus_{\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\in\mathcal{C}_4} L_{4/5}(\alpha_1) \otimes L_{4/5}(\alpha_2) \otimes L_{4/5}(\alpha_3) \otimes L_{4/5}(\alpha_4), \qquad (3.13)$$

where $L_{4/5}(0) = L(^4\!/_5, 0) \oplus L(^4\!/_5, 3) = \mathcal{W}(^4\!/_5)$ and $L_{4/5}(\pm 1) = L(^4\!/_5, ^2\!/_3)^{\pm}$ as $\mathcal{W}(^4\!/_5)$-modules. Note also that the cosets of L_{3A} in E_6 can be parameterized by the ternary repetition code of length 3:

$$\mathcal{D} = \{(0,0,0),\ (1,1,1),\ (-1,-1,-1)\}. \tag{3.14}$$

Thus the commutant subalgebra $U_{F(3A)}$ is the ternary code VOA $M_{\mathcal{D}}$ defined in [KMY]:

$$M_{\mathcal{D}} = (L(^4\!/_5, 0) \oplus L(^4\!/_5, 3))^{\otimes 3} \oplus \left(L(^4\!/_5, ^2\!/_3)^{+}\right)^{\otimes 3} \oplus \left(L(^4\!/_5, ^2\!/_3)^{-}\right)^{\otimes 3}. \tag{3.15}$$

It is shown in [KMY] that

$$\ker_{V_{\sqrt{2}A_2}}(\omega_{A_2} - \tilde{\omega}_{A_2})_{(0)} \simeq \mathcal{W}(^4\!/_5) = L(^4\!/_5, 0) \oplus L(^4\!/_5, 3)$$

and we get:

Lemma 3.6. *One has a decomposition*

$$U_{F(3A)} \ \simeq\ \mathcal{W}(^4\!/_5)^{\otimes 3} \ \oplus\ \left(L(^4\!/_5, ^2\!/_3)^{+}\right)^{\otimes 3} \ \oplus\ \left(L(^4\!/_5, ^2\!/_3)^{-}\right)^{\otimes 3}$$

as a $\mathcal{W}(^4\!/_5)^{\otimes 3}$-module.

Now we see that the weight two subspace of $U_{F(3A)}$ is 5-dimensional and coincides with $\mathcal{G}_{F(3A)}$.

Remark 3.7. By the comments after Remark 3.3, we know that $\mathcal{F}_{3A}$ acts on $V_{F(3A)}$. In this case, the fixed point space is $V_{F(3A)}^{\mathcal{F}_{3A}} \simeq L_{4/5}(0) \otimes L_{4/5}(0) \otimes L_{4/5}(0) \otimes L_{4/5}(0) \simeq \mathcal{W}(^4\!/_5)^{\otimes 4}$ while $L_{4/5}(\alpha_1) \otimes L_{4/5}(\alpha_2) \otimes L_{4/5}(\alpha_3) \otimes L_{4/5}(\alpha_4)$, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathcal{C}_4$, are character spaces of $\mathcal{F}_{3A}$ in $V_{F(3A)}$. The automorphism group of $V_{F(3A)} \simeq M_{\mathcal{C}_4}$ was computed in [KMY, Proposition 5.4]. It is isomorphic to $3^2.(2\mathrm{S}_4) \simeq 3^2.\mathrm{Aut}(\mathcal{C}_4) \simeq \mathrm{AGL}_2(3)$. The subgroup $\mathcal{F}_{3A} : \langle \tau_{\hat{e}} \rangle \simeq 3^2 : 2$ fixes the four $c = 4/5$ Virasoro vectors and is the stabilizer of the sub-VOA $V_{F(3A)}^{\mathcal{F}_{3A}} \simeq L_{4/5}(0) \otimes L_{4/5}(0) \otimes L_{4/5}(0) \otimes L_{4/5}(0) \simeq \mathcal{W}(^4\!/_5)^{\otimes 4}$.

In the Appendix, we will need a result about generators for $V_{F(3A)}$.

Lemma 3.8. *Let ρ_1 and ρ_2 be generating $\mathcal{F}_{3A} \simeq 3^2$. Then $V_{F(3A)}$ is generated by $\hat{e}$, $\rho_1\hat{e}$ and $\rho_2\hat{e}$.*

Proof: By Remark 3.7, we know that $\mathcal{F}_{3A}$ acts on $V_{F(3A)} < V_{\sqrt{2}E_8}$ and the fixed point subspace is $V_{F(3A)}^{\mathcal{F}_{3A}} \simeq \mathcal{W}(^4\!/_5)^{\otimes 4}$. It is clear that $e^{i,j} := \rho_1^i \rho_2^j \hat{e}$, $0 \le i,\ j \le 2$, are contained in $V_{F(3A)}$ since $\hat{e} \in V_{F(3A)}$.

Let W be the sub-VOA of $V_{F(3A)}$ generated by $\{e^{i,j} \mid 0 \le i,\ j \le 2\}$. For each i, $j = 0,\ 1,\ 2$ with $(i,j) \ne (0,0)$, the Ising vectors $e^{0,0}$ and $e^{i,j}$ generate a sub-VOA in $V_{F(3A)}$ isomorphic to U_{3A} which contains a subalgebra isomorphic to $\mathcal{W}(^4\!/_5)$ fixed by $\mathcal{F}_{3A}$

(cf. [LYY2]). In fact, the W_3-algebra $\mathcal{W}(4/5)$ is contained in a lattice sub-VOA $V_{\sqrt{2}A_2} < V_{\sqrt{2}E_8}$, where $\sqrt{2}A_2 < \sqrt{2}A_2^4 < \sqrt{2}E_8$. By varying i, j (say, $(i,j) = (1,0)$, $(0,1)$, $(1,1)$ and $(1,2)$), one can obtain four mutually orthogonal $\mathcal{W}(4/5)$ in $V_{F(3A)}^{\mathcal{F}_{3A}}$. Thus we have $V_{F(3A)}^{\mathcal{F}_{3A}} \simeq \mathcal{W}(4/5)^{\otimes 4} < W$.

Now let $\zeta = e^{2\pi i/3}$ and let

$$v_{m,n} = \sum_{i=0}^{2}\sum_{j=0}^{2} \zeta^{mi+nj}\, e^{-i,-j},$$

for any m, $n = 0$, 1, 2. Then by direct calculation, it is easy to verify that

$$\rho_1^k \rho_2^\ell (v_{m,n}) = \zeta^{km+\ell n} v_{m,n}.$$

In other words, $v_{m,n}$ spans a 1-dimensional $\mathcal{F}_{3A}$-submodule affording the character $\chi_{m,n}$, where $\chi_{m,n}(\rho_1^i\rho_2^j) = \zeta^{mi+nj}$. It also generates the irreducible $V_{F(3A)}^{\mathcal{F}_{3A}} \simeq \mathcal{W}(4/5)^{\otimes 4}$-submodule affording the character $\chi_{m,n}$. Hence, W contains a sub-VOA isomorphic to $M_{\mathcal{C}} \simeq V_{F(3A)}$ and we have $W = V_{F(3A)}$.

Now note that the 3A-algebra generated by $e^{0,0}$ and $e^{1,0}$ contains $e^{2,0} = \tau_{e^{0,0}} e^{1,0}$. Similarly, we get $e^{0,2}$. Since ρ_1 and ρ_2 are inverted by $\tau_{\hat{e}}$, we have

$$\tau_{e^{i,0}}(e^{0,j}) = \tau_{\rho_1^i \hat{e}}(\rho_2^j \hat{e}) = \rho_1^i \tau_{\hat{e}} \rho_1^{-i} \rho_2^j \hat{e} = \rho_1^{2i}\rho_2^{2j}\hat{e} = e^{2i,2j}$$

and thus $e^{2i,2j}$ is contained in the 3A-algebra generated by $e^{i,0}$ and $e^{0,j}$. Hence, $V_{F(3A)}$ is generated by $e^{0,0} = \hat{e}$, $\rho_1\hat{e} = e^{1,0}$ and $\rho_2\hat{e} = e^{0,1}$. ∎

Remark 3.9. The irreducible modules for ternary code VOA M_D have been studied in [La]. It is known (cf. Theorem 4.8 and 4.10 of [La]) that if D is a self-dual ternary code, then all irreducible M_D-modules can be realized (using coset or GKO construction) as submodules of a certain lattice VOA V_{Γ_D}. We refer to Section 2 of [KMY] or Section 3.2 of [La] for the precise definition of Γ_D. When $D = \mathcal{C}_4$ is the tetra code, we have $\Gamma_D = \Gamma_{\mathcal{C}_4} \simeq \sqrt{2}E_8$ and all irreducible $M_{\mathcal{C}_4}$-modules are contained in the lattice VOA $V_{\sqrt{2}E_8}$. The subgroup $3^2{:}2 \simeq \mathcal{F}_{3A}{:}\langle \tau_{\hat{e}}\rangle < \mathrm{Aut}(V_{\sqrt{2}E_8})$ actually acts faithfully on all irreducible $M_{\mathcal{C}_4}$-submodules. Thus, if V is a VOA containing $V_{F(3A)} \simeq M_{\mathcal{C}_4}$, then the involutions $\tau_{\hat{e}}$, $\tau_{\rho_1\hat{e}}$ and $\tau_{\rho_2\hat{e}}$ generate a group of the shape $3^2{:}2$ in $\mathrm{Aut}(V)$.

Subalgebras generated by $\tilde{v}$ and $\tilde{v}'$. Set

$$\tilde{v} := \tilde{\omega}_{E_6} \qquad \text{and} \qquad \tilde{v}' := \rho_{nX}\tilde{\omega}_{E_6}. \tag{3.16}$$

By definition, the Virasoro vectors $\tilde{v}$ and $\tilde{v}'$ are contained in $U_{F(nX)}$. We will discuss $\mathcal{G}_{F(nX)}$ and the subalgebras generated by $\tilde{v}$ and $\tilde{v}'$.

1A case. In this case $L_{1A} \simeq E_6$ and ρ_{1A} is trivial. Thus $\tilde{v}' = \tilde{v}$, $\langle \tilde{v}, \tilde{v}' \rangle = 3/7$ and $\tilde{v}$ generates $\mathcal{G}_{F(1A)}$, but not $U_{F(1A)}$.

2A case. In this case $L_{2A} \simeq A_1 \oplus A_5$ and $\ell = 2$.

The vectors ω^1 and ω^2 are Virasoro vectors with central charges 1/2 and 5/4, respectively, and $X = X^1$ is a highest weight vector for $\mathrm{Vir}(\omega^1) \otimes \mathrm{Vir}(\omega^2)$ with highest weight $(1/2, 3/2)$. One easily obtains:

Lemma 3.10. *The Griess algebra $\mathcal{G}_{F(2A)}$ is spanned by ω^1, ω^2, and X and we have the following commutative algebra structure on $\mathcal{G}_{F(2A)}$:*

$a_{(1)}b$	ω^1	ω^2	X
ω^1	$2\omega^1$	0	$\frac{1}{2}X$
ω^2		$2\omega^2$	$\frac{3}{2}X$
X			$80\omega^1 + 96\omega^2$

$\langle a, b \rangle$	ω^1	ω^2	X
ω^1	$\frac{1}{4}$	0	0
ω^2		$\frac{5}{8}$	0
X			40

One verifies that

$$\tilde{v} = \frac{2}{7}\omega^1 + \frac{4}{7}\omega^2 + \frac{1}{14}X, \quad \tilde{v}' = \frac{2}{7}\omega^1 + \frac{4}{7}\omega^2 - \frac{1}{14}X, \quad \langle \tilde{v}, \tilde{v}' \rangle = \frac{1}{49}.$$

It is also easily verified that $\mathcal{G}_{F(2A)}$ is generated by $\tilde{v}$ and $\tilde{v}'$. Set

$$u = \frac{5}{7}\omega^1 + \frac{3}{7}\omega^2 - \frac{1}{14}X.$$

Then $\tilde{v}$ and u are the mutually orthogonal Virasoro vectors with central charges 6/7 and 25/28, respectively, used in the decomposition of $U_{F(2A)}$ given before.

3A case. In this case $L_{3A} \simeq A_2 \oplus A_2 \oplus A_2$ and $\ell = 3$.

The three vectors ω^1, ω^2 and ω^3 are mutually orthogonal Virasoro vectors with central charge 4/5, and X^1, X^2 are highest weight vectors for $\mathrm{Vir}(\omega^1) \otimes \mathrm{Vir}(\omega^2) \otimes \mathrm{Vir}(\omega^3)$ with highest weight $(2/3, 2/3, 2/3)$. Again, the following result is easily obtained:

Lemma 3.11. *The Griess algebra $\mathcal{G}_{F(3A)}$ is spanned by ω^1, ω^2, ω^3, X^1 and X^2. Moreover, we have the following commutative algebra structure on $\mathcal{G}_{F(3A)}$:*

$a_{(1)}b$	ω^1	ω^2	ω^3	X^1	X^2
ω^1	$2\omega^1$	0	0	$\frac{2}{3}X^1$	$\frac{2}{3}X^2$
ω^2		$2\omega^2$	0	$\frac{2}{3}X^1$	$\frac{2}{3}X^2$
ω^3			$2\omega^3$	$\frac{2}{3}X^1$	$\frac{2}{3}X^2$
X^1				$8X^2$	$45(\omega^1 + \omega^2 + \omega^3)$
X^2					$8X^1$

$\langle a, b \rangle$	ω^1	ω^2	ω^3	X^1	X^2
ω^1	$\frac{2}{5}$	0	0	0	0
ω^2		$\frac{2}{5}$	0	0	0
ω^3			$\frac{2}{5}$	0	0
X^1				0	27
X^2					0

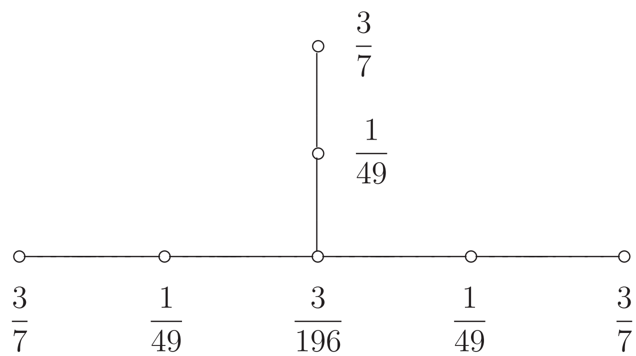
One verifies that

$$\tilde{v} = \frac{5}{14}(\omega^1+\omega^2+\omega^3) + \frac{1}{14}X^1 + \frac{1}{14}X^2, \qquad \langle \tilde{v}, \tilde{v}' \rangle = \frac{3}{196}, \qquad \text{where } \zeta = e^{2\pi\sqrt{-1}/3}.$$
$$\tilde{v}' = \frac{5}{14}(\omega^1+\omega^2+\omega^3) + \frac{\zeta}{14}X^1 + \frac{\zeta^{-1}}{14}X^2,$$

In this case, the Griess algebra $\mathcal{G}_{F(3A)}$ is not generated by $\tilde{v}$ and $\tilde{v}'$. Let ν be the diagram automorphism of the affine E_6 diagram of order 3 defined as: $\alpha_0 \mapsto \alpha_1 \mapsto \alpha_5 \mapsto \alpha_0$, $\alpha_6 \mapsto \alpha_2 \mapsto \alpha_4 \mapsto \alpha_6$ and $\alpha_3 \mapsto \alpha_3$ on the diagram (3.4). Since $\sqrt{2}E_6$ is doubly even, we have a splitting $\mathrm{Aut}(V_{\sqrt{2}E_6}) \simeq \mathrm{Hom}_{\mathbb{Z}}(E_6, \mathbb{C}^*) \rtimes \mathrm{Aut}(E_6)$ (see Theorem 2.1 of [DN] and Chapter 5 of [FLM]). Then ν canonically acts on the Griess subalgebra above and we find that $\tilde{v}$ and $\tilde{v}'$ generate the fixed point subalgebra

$$\mathcal{G}_{F(3A)}^{\langle\nu\rangle} = \mathrm{span}_{\mathbb{C}}\{\omega^1+\omega^2+\omega^3, X^1, X^2\}.$$

Summarizing, we have obtained the following table of values of inner products between $\tilde{v}$ and $\tilde{v}'$:



Remark 3.12. By the computation above, we see that in the 3A case the Griess subalgebra generated by $\tilde{v}$ and $\tilde{v}'$ coincides with the fixed point subalgebra $\mathcal{G}_{F(3A)}^{\langle\nu\rangle}$. This is the only case where the corresponding node is fixed by the diagram automorphism.

4 The 3A-algebra for the Monster

In this section, we will review and list some properties of a VOA called the 3A-algebra for the Monster which is related to certain dihedral groups of order 6 in the Monster (cf. [LYY1, LYY2, S]). By using the VOA structure of the 3A-algebra, we will show in Theorem 4.10 that certain commutant algebras of the Virasoro VOA $L(4/5, 0)$ in an arbitrary VOA, satisfying few mild assumptions, have a subgroup of automorphisms satisfying the 3-transposition property. These results will be used in the last section to study the Moonshine VOA and its subalgebra related to the Fischer group.

We first consider the extended simple Virasoro VOAs $\mathcal{W}(4/5) = L(4/5, 0) \oplus L(4/5, 3)$ and $\mathcal{W}(6/7) = L(6/7, 0) \oplus L(6/7, 5)$ in Theorem 2.7. It is discussed in [LYY2, Mi3, SY] that the Moonshine VOA contains the following subalgebra, which is a simple current extension of $\mathcal{W}(4/5) \otimes \mathcal{W}(6/7)$:

$$\begin{aligned} U_{3A} := \ & \big(L(4/5,0) \oplus L(4/5,3)\big) \otimes \big(L(6/7,0) \oplus L(6/7,5)\big) \\ & \oplus\ L(4/5,2/3)^+ \otimes L(6/7,4/3)^+ \ \oplus\ L(4/5,2/3)^- \otimes L(6/7,4/3)^-. \end{aligned} \tag{4.1}$$

Moreover, a dihedral group of order 6 can be defined using U_{3A} such that all order 3 elements are in the 3A conjugacy class (loc. cit.). Therefore, U_{3A} is closely related to the 3A-element of the Monster and we will call it the *3A-algebra* for the Monster.

Remark 4.1. The 3A-algebra can be also constructed along the recipe described in Section 3 via the embedding $A_2 \oplus E_6 \hookrightarrow E_8$, which corresponds to the 3A node of the McKay E_8-observation [LYY1].

In Sections 4.1 and 4.2, we will review the results obtained in [LYY2, SY] which we will use in Section 4.3 to prove the 3-transposition property of Miyamoto involutions associated to derived $c = 6/7$ Virasoro vectors (cf. Theorem 4.10).

4.1 Griess algebra

In this subsection, we will recall some basic properties of the 3A-algebra U_{3A} from [LYY2, SY]. The Griess algebra of U_{3A} is 4-dimensional and can be described as follows.

Let ω^1 and ω^2 be the Virasoro vectors of the subalgebras $L(4/5, 0)$ and $L(6/7, 0)$ of U_{3A} in (4.1), respectively, and let $X^\pm$ be the highest weight vectors of the components $L(4/5, 2/3)^\pm \otimes L(6/7, 4/3)^\pm$ of U_{3A}.

Lemma 4.2 ([LYY2])**.** *The commutative algebra structure on the Griess algebra of U_{3A} is given by:*

$a_{(1)}b$	ω^1	ω^2	X^+	X^-
ω^1	$2\omega^1$	0	$\frac{2}{3}X^+$	$\frac{2}{3}X^-$
ω^2		$2\omega^2$	$\frac{4}{3}X^+$	$\frac{4}{3}X^-$
X^+			$20X^-$	$135\omega^1 + 252\omega^2$
X^-				$20X^+$

$\langle a, b\rangle$	ω^1	ω^2	X^+	X^-
ω^1	$\frac{2}{5}$	0	0	0
ω^2		$\frac{3}{7}$	0	0
X^+			0	81
X^-				0

The Virasoro vectors of U_{3A} are classified in [SY] and there are in total three Ising vectors in U_{3A}. Let ζ be a primitive cubic root of unity. Then

$$e^i = \frac{5}{32}\omega^1 + \frac{7}{16}\omega^2 + \frac{1}{32}\zeta^i X^+ + \frac{1}{32}\zeta^{-i} X^-, \quad i = 0,\, 1,\, 2, \tag{4.2}$$

provide all the Ising vectors of U_{3A}. The associated τ-involutions satisfy $|\tau_{e^i}\tau_{e^j}| = 3$ if $i \neq j$ and therefore they generate the symmetric group S_3 in $\mathrm{Aut}(U_{3A})$. Indeed, it is known [LLY, Mi3, SY] that $\tau_{e^i}\tau_{e^j}$ coincides with the order three elements ξ or ξ^{-1} in Theorem 2.13. The VOA U_{3A} is generated by any two of these Ising vectors and correspondingly $\mathrm{Aut}(U_{3A}) \simeq \mathrm{S}_3$ is also generated by the associated τ-involutions.

It is shown in [SY] that U_{3A} has exactly four simple Virasoro vectors with central charge $4/5$, namely, ω^1 and the following three vectors:

$$x^i = \frac{1}{16}\omega^1 + \frac{7}{8}\omega^2 - \frac{1}{48}\zeta^i X^+ - \frac{1}{48}\zeta^{-i} X^-, \quad i = 0,\, 1,\, 2. \tag{4.3}$$

Among these four vectors, only ω^1 is characteristic in the sense it is fixed by $\mathrm{Aut}(U_{3A})$, whereas the other three vectors are conjugated by τ-involutions τ_{e^i}, $i = 0, 1, 2$. We call $\omega^1 + \omega^2$ the *characteristic Virasoro frame* of U_{3A}. By (4.1), we see that ω^1 and ω^2 are extendable. Here we show that ω^1 is the unique extendable simple $c = 4/5$ Virasoro vector of U_{3A}.

Lemma 4.3 ([LYY2]). *Let y be one of the x^i, $i = 0, 1, 2$. Then as a module over $\mathrm{Vir}(y) \simeq L(4/5, 0)$, we have $U_{3A} = (U_{3A})_y[0] \oplus (U_{3A})_y[3] \oplus (U_{3A})_y[2/3] \oplus (U_{3A})_y[1/8] \oplus (U_{3A})_y[13/8]$. Moreover, $(U_{3A})_2 \cap (U_{3A})_y[13/8] \neq 0$.*

By Theorem 2.11, there is no untwisted $\mathcal{W}(4/5)$-module which contains an $L(4/5, 0)$-submodule isomorphic to $L(4/5, 13/8)$, and therefore we see that the $c = 4/5$ Virasoro vectors x^i, $i = 0, 1, 2$, are not extendable.

4.2 Representation theory

The representation theory of U_{3A} was completed in [SY].

Theorem 4.4 ([SY]). *The VOA U_{3A} is rational and there are six isomorphism types of irreducible modules over U_{3A} with the following shapes as $\mathrm{Vir}(\omega^1) \otimes \mathrm{Vir}(\omega^2)$-modules:*

$$\begin{aligned}
U(0) &= U_{3A} = [0, 0] \oplus [3, 0] \oplus [0, 5] \oplus [3, 5] \oplus 2\,[2/3, 4/3],\\
U(1/7) &= [0, 1/7] \oplus [0, 22/7] \oplus [3, 1/7] \oplus [3, 22/7] \oplus 2\,[2/3, 10/21],\\
U(5/7) &= [0, 5/7] \oplus [0, 12/7] \oplus [3, 5/7] \oplus [3, 12/7] \oplus 2\,[2/3, 1/21],\\
U(2/5) &= [2/5, 0] \oplus [2/5, 5] \oplus [7/5, 0] \oplus [7/5, 5] \oplus 2\,[1/15, 4/3],\\
U(19/35) &= [2/5, 1/7] \oplus [2/5, 22/7] \oplus [7/5, 1/7] \oplus [7/5, 22/7] \oplus 2\,[1/15, 10/21],\\
U(4/35) &= [2/5, 5/7] \oplus [2/5, 12/7] \oplus [7/5, 5/7] \oplus [7/5, 12/7] \oplus 2\,[1/15, 1/21],
\end{aligned}$$

where $[h_1, h_2]$ *denotes an irreducible* $\mathrm{Vir}(\omega^1) \otimes \mathrm{Vir}(\omega^2) \simeq L(4/5, 0) \otimes L(6/7, 0)$*-module isomorphic to* $L(4/5, h_1) \otimes L(6/7, h_2)$*.*

By the list of irreducible modules above, we see that U_{3A} is a maximal extension of $L(4/5, 0) \otimes L(6/7, 0)$ as a simple VOA. We remark the following fundamental observation.

Lemma 4.5. *Let* V *be a VOA and* $e \in V$ *be an Ising vector. Then any* V*-module* M *is* τ_e*-stable, that is, the* τ_e*-conjugated module* $\tau_e \circ M$ *is isomorphic to* M *itself. In particular, if* G *is a subgroup of* $\mathrm{Aut}(V)$ *generated by* τ*-involutions associated to Ising vectors of* V *then* M *is* G*-stable, that is,* $g \circ M \simeq M$ *for all* $g \in G$*.*

As we discussed, $\mathrm{Aut}(U_{3A}) \simeq \mathrm{S}_3$ is generated by τ-involutions associated to Ising vectors of U_{3A} and therefore all irreducible U_{3A}-modules are S_3-invariant. In general, if an irreducible V-module M is G-stable then we have a projective action of G on M (cf. [DY]). But in our case, we have an *ordinary* action of S_3 on each irreducible U_{3A}-module. For, we can find all the irreducible U_{3A}-modules as a submodule of a larger VOA, say $V_{\sqrt{2}E_8}$ for example, on which we have an ordinary S_3-action (cf. [LY, LYY2]; consider $U_{3A}^{\mathrm{S}_3} \subset (V_{\sqrt{2}A_2} \otimes V_{\sqrt{2}E_6})^+ \subset V_{\sqrt{2}E_8}^+$). Let M_0, M_1 and M_2 be the principal, signature and 2-dimensional irreducible representations of S_3. As a $U_{3A}^{\mathrm{S}_3} \otimes \mathbb{C}\mathrm{S}_3$-module, one has the following decompositions:[2]

$$
\begin{aligned}
U(0) &= \big([0,0] \oplus [3,5]\big) \otimes M_0 \;\oplus\; \big([3,0] \oplus [0,5]\big) \otimes M_1 \;\oplus\; [2/3, 4/3] \otimes M_2 \\
U(1/7) &= \big([0,22/7] \oplus [3,22/7]\big) \otimes M_0 \;\oplus\; \big([0,1/7] \oplus [3,1/7]\big) \otimes M_1 \;\oplus\; [2/3, 10/21] \otimes M_2, \\
U(5/7) &= \big([0,5/7] \oplus [3,5/7]\big) \otimes M_0 \;\oplus\; \big([0,12/7] \oplus [3,12/7]\big) \otimes M_1 \;\oplus\; [2/3, 1/21] \otimes M_2, \\
U(2/5) &= \big([2/5,0] \oplus [7/5,0]\big) \otimes M_0 \;\oplus\; \big([2/5,5] \oplus [7/5,5]\big) \otimes M_1 \;\oplus\; [1/15, 4/3] \otimes M_2, \\
U(19/35) &= \big([2/5,22/7] \oplus [7/5,22/7]\big) \otimes M_0 \;\oplus\; \big([2/5,1/7] \oplus [7/5,1/7]\big) \otimes M_1 \;\oplus\; [1/15, 10/21] \otimes M_2, \\
U(4/35) &= \big([2/5,5/7] \oplus [7/5,5/7]\big) \otimes M_0 \;\oplus\; \big([2/5,12/7] \oplus [7/5,12/7]\big) \otimes M_1 \;\oplus\; [1/15, 1/21] \otimes M_2,
\end{aligned}
\tag{4.4}
$$

where $[h_1, h_2]$ denotes a $\mathrm{Vir}(\omega^1) \otimes \mathrm{Vir}(\omega^2)$-module isomorphic to $L(4/5, h_1) \otimes L(6/7, h_2)$.

4.3 3-transposition property of σ-involutions

We consider involutions induced by the 3A-algebra. We will again refer to Section 2.1 for the definition of simple $c = 6/7$ Virasoro vectors of σ-type and their corresponding σ-involutions. See Lemma 2.4 and Eq. (2.4) for the details.

Let V be a VOA and let $u \in V$ be a simple $c = 4/5$ Virasoro vector.

[2] There is an Ising vector $e \in V_{\sqrt{2}E_8}$ such that τ_e acts by -1 on the weight one subspace. Then considering an embedding $U_{3A}^{\mathrm{S}_3} \subset (V_{\sqrt{2}A_2} \otimes V_{\sqrt{2}E_6})^+ \subset V_{\sqrt{2}E_8}^+$ with $e \in U_{3A}$ as in [LYY2], one can verify the decomposition.

Definition 4.6. A simple $c = 6/7$ Virasoro vector $v \in \mathrm{Com}_V(\mathrm{Vir}(u))$ is called *a derived Virasoro vector with respect to* u if there exists a sub-VOA U of V isomorphic to the 3A-algebra U_{3A} such that $u + v$ is the characteristic Virasoro frame of U.

Lemma 4.7. *A derived* $c = 6/7$ *Virasoro vector* $v \in \mathrm{Com}_V(\mathrm{Vir}(u))$ *with respect to* u *is of* σ*-type on the commutant* $\mathrm{Com}_V(\mathrm{Vir}(u))$.

Proof: Assume that V contains a subalgebra U isomorphic to the 3A-algebra U_{3A} as in Definition 4.6. For an irreducible U-module M, we denote

$$H_M := \mathrm{Hom}_{U_{3A}}(M, V).$$

Then we have the isotypical decomposition

$$V = \bigoplus_{M \in \mathrm{Irr}(U_{3A})} M \otimes H_M.$$

By definition, $u + v$ is the characteristic Virasoro frame of U_{3A}. Consider V as a module over its subalgebra $\mathrm{Vir}(u) \otimes \mathrm{Vir}(v) \otimes \mathrm{Com}_V(U)$. By Theorem 4.4 we have the following decomposition:

$$\begin{aligned} &V_u[0] = \mathrm{Vir}(u) \otimes \mathrm{Com}_V(\mathrm{Vir}(u)), \\ &\mathrm{Com}_V(\mathrm{Vir}(u)) = [0 \oplus 5] \otimes H_{U(0)} \oplus [1/7 \oplus 22/7] \otimes H_{U(1/7)} \oplus [5/7 \oplus 12/7] \otimes H_{U(5/7)}, \end{aligned} \tag{4.5}$$

where $[h_1 \oplus h_2]$ denotes an irreducible $\mathcal{W}(6/7)$-module isomorphic to $L(6/7, h_1) \oplus L(6/7, h_2)$. By the decomposition above, we see that v is of σ-type on $\mathrm{Com}_V(\mathrm{Vir}(u))$. ∎

We consider the one-point stabilizer

$$\mathrm{Stab}_{\mathrm{Aut}(V)}(u) := \{h \in \mathrm{Aut}(V) \mid hu = u\}. \tag{4.6}$$

Each $h \in \mathrm{Stab}_{\mathrm{Aut}(V)}(u)$ keeps the isotypical component $V_u[h]$ invariant so that by restriction we can define a group homomorphism

$$\begin{array}{rccc} \psi_u : & \mathrm{Stab}_{\mathrm{Aut}(V)}(u) & \longrightarrow & \mathrm{Aut}(\mathrm{Com}_V(\mathrm{Vir}(u))), \\ & h & \longmapsto & h|_{\mathrm{Com}_V(\mathrm{Vir}(u))}. \end{array} \tag{4.7}$$

Let $v \in \mathrm{Com}_V(\mathrm{Vir}(u))$ be a derived $c = 6/7$ Virasoro vector with respect to u. By Lemmas 4.7 and 2.4, we have an involution $\sigma_v \in \mathrm{Aut}(\mathrm{Com}_V(\mathrm{Vir}(u)))$. Now let e be an Ising vector of U. By Lemma 4.5 and Eq. (4.4), we see that τ_e keeps $\mathrm{Com}_V(\mathrm{Vir}(u))$ invariant and, in fact, we have:

Lemma 4.8. $\psi_u(\tau_e) = \sigma_v$ *for any Ising vector* $e \in U$.

Let J be the set of all derived $c = 6/7$ Virasoro vectors of $\mathrm{Com}_V(\mathrm{Vir}(u))$. We will prove that the set of involutions

$$\{\sigma_v \in \mathrm{Aut}(\mathrm{Com}_V(\mathrm{Vir}(e))) \mid v \in J\}$$

satisfies a 3-transposition property.

We say a VOA W over $\mathbb{R}$ is *compact* if W has a positive definite invariant bilinear form. A real sub-VOA W is said to be a *compact real form* of a VOA V over $\mathbb{C}$ if W is compact and $V \simeq \mathbb{C} \otimes_{\mathbb{R}} W$.

We recall the following interesting theorem of Sakuma.

Theorem 4.9 ([S]). *Let W be a VOA over $\mathbb{R}$ with grading $W = \bigoplus_{n\geq 0} W_n$, $W_0 = \mathbb{R}\mathbb{1}$ and $W_1 = 0$, and assume W is compact, that is, the normalized invariant bilinear form on W is positive definite. Let x, y be Ising vectors in W and denote by $U(x,y)$ the subalgebra of W generated by x and y. Then:*
(1) The 6-transposition property $|\tau_x\tau_y| \leq 6$ holds on W.
(2) There are exactly nine possible inequivalent structures of the Griess algebra on the weight two subspace $U(x,y)_2$ of $U(x,y)$.
(3) The Griess algebra structure on $U(x,y)_2$ is unique if $|\tau_x\tau_y| = 6$ and in this case $U(x,y)$ is a copy of the 6A-algebra.

The following is the main theorem of this section.

Theorem 4.10. *Suppose that $V_1 = 0$ and V has a compact real form $V_{\mathbb{R}}$ and every Ising vector of V is in $V_{\mathbb{R}}$. Then for any v^1, $v^2 \in J$, we have $|\sigma_{v^1}\sigma_{v^2}| \leq 3$ on* $\mathrm{Com}_V(\mathrm{Vir}(u))$.

Proof: Let v^1, $v^2 \in J$. Then there exist subalgebras U^1 and U^2 of V isomorphic to the 3A-algebra such that u is a simple extendable $c = 4/5$ Virasoro vector in $U^1 \cap U^2$ and $v^1 \in \mathrm{Com}_{U^1}(\mathrm{Vir}(u))$, $v^2 \in \mathrm{Com}_{U^2}(\mathrm{Vir}(u))$. Let a, a', a'' be the three distinct Ising vectors in U^1 and b, b', b'' be the three distinct Ising vectors in U^2. Set $g := \tau_a\tau_{a'}$. Then g is an order three element induced by the extended Virasoro VOA $\mathcal{W}(4/5)$ and we have $a' = ga$ and $a'' = g^2a$. By our settings, $\tau_b\tau_{b'} = g$ or g^2 so that we may also assume $g = \tau_b\tau_{b'}$, $b' = gb$ and $b'' = g^2b$. Note that $\psi_u(g) = 1$.

Now, recall that

$$\sigma_{v^1} = \psi_u(\tau_a) = \psi_u(\tau_{a'}) = \psi_u(\tau_{a''}) \quad \text{and} \quad \sigma_{v^2} = \psi_u(\tau_b) = \psi_u(\tau_{b'}) = \psi_u(\tau_{b''}).$$

Thus, the order of $\sigma_{v^1}\sigma_{v^2} = \psi_u(\tau_a\tau_b)$ must divide that of $\tau_a\tau_b$.

By Sakuma's Theorem 4.9, we know that $|\tau_a\tau_b| \leq 6$. If $|\tau_a\tau_b| \leq 3$, there is nothing to prove. So we assume $|\tau_a\tau_b| = 4$, 5 or 6.

First, we will note that g commutes with $\tau_a\tau_b$ since $\tau_a g = g^{-1}\tau_a$ and $\tau_b g = g^{-1}\tau_b$.

Case $|\tau_a\tau_b| = 4$: In this case, $\tau_a\tau_b g$ has order 12, which is impossible since $\tau_a\tau_b g = \tau_a\tau_b(\tau_b\tau_{b'}) = \tau_a\tau_{b'}$ has order ≤ 6.

Case $|\tau_a\tau_b| = 5$: In this case, $\tau_a\tau_b g = \tau_a\tau_{b'}$ has order 15, which is again impossible.

Case $|\tau_a\tau_b| = 6$: In this case, $\tau_a\tau_b g$ and $\tau_a\tau_b g^2$ have order 6 or 2.

Claim: If $|\tau_a\tau_b g| = 6$ then $|\tau_a\tau_b g^2| = 2$.

The reason is as follows. Suppose both of them have order 6. Since $\tau_a\tau_b g = \tau_a\tau_{b'}$ and $\tau_a\tau_b g^2 = \tau_a\tau_b(\tau_{b'}\tau_b) = \tau_a\tau_{\tau_b b'} = \tau_a\tau_{b''}$, we have $\langle a, b\rangle = \langle a, b'\rangle = \langle a, b''\rangle = 5/2^{10}$ and hence $\langle g^i a, g^j b\rangle = 5/2^{10}$ for all i, $j = 0, 1, 2$ (cf. the Appendix in [HLY]).

Now by using the structure of the 3A-algebra U_{3A}, we may write the Ising vectors a and b as

$$\begin{aligned} a &= \frac{5}{32}u + \frac{7}{16}v^1 + \frac{1}{32}(X^1 + X^2),\\ b &= \frac{5}{32}u + \frac{7}{16}v^2 + \frac{1}{32}(Y^1 + Y^2), \end{aligned}$$

where X^1, X^2 (resp. Y^1, Y^2) are certain highest weight vectors of weight $(2/3, 4/3)$ with respect to $\mathrm{Vir}(u) \otimes \mathrm{Vir}(v^1)$ (resp. $\mathrm{Vir}(u) \otimes \mathrm{Vir}(v^2)$). Using (4.2), we get

$$\begin{aligned} a + ga + g^2 a = a + a' + a'' &= \frac{3}{32}(5u + 14v^1),\\ b + gb + g^2 b = b + b' + b'' &= \frac{3}{32}(5u + 14v^2). \end{aligned}$$

Thus, we have

$$\left\langle a + ga + g^2 a,\, b + gb + g^2 b\right\rangle = \frac{9}{2^{10}}\left\langle 5u + 14v^1, 5u + 14v^2\right\rangle.$$

Since $\langle u, u\rangle = 2/5$ and $\langle u, v^i\rangle = 0$ for $i = 1, 2$ by Lemma 4.2, this implies

$$\frac{5}{2^{10}} = \frac{5^2}{2^{10}}\langle u, u\rangle + \frac{7^2}{2^8}\langle v^1, v^2\rangle = \frac{5}{2^9} + \frac{7^2}{2^8}\langle v^1, v^2\rangle$$

and thus $\langle v^1, v^2\rangle = -5/196 < 0$. This is impossible by the Norton inequality

$$\langle v^1_{(1)}v^1, v^2_{(1)}v^2\rangle \geq \langle v^1_{(1)}v^2, v^1_{(1)}v^2\rangle \geq 0$$

(cf. Theorem 6.3 and Lemma 6.5 in [Mi1], see also [B][3]) and the claim follows.

Therefore, $\tau_a\tau_b g = \tau_a\tau_{b'}$ or $\tau_a\tau_b g^2 = \tau_a\tau_{b''}$ has order 2 and hence

$$\sigma_{v^1}\sigma_{v^2} = \psi_u(\tau_a\tau_b) = \psi_u(\tau_a\tau_{b'}) = \psi_u(\tau_a\tau_{b''})$$

is of order at most 2. ∎

[3]A sketch of proof is given in [B].

5 The Fischer group

In this section, we will discuss the properties of the commutant vertex operator subalgebra $V\!F^{\natural}$ of the Moonshine VOA $V^{\natural}$. We will show that the full Fischer 3-transposition group Fi_{24} is a subgroup of $\mathrm{Aut}(V\!F^{\natural})$. We also show that there exist one-to-one correspondences between 2C-involutions of Fi_{24} and derived $c = 6/7$ Virasoro vectors in $V\!F^{\natural}$.

Finally, we will discuss the embeddings of $U_{F(nX)}$ into $V\!F^{\natural}$. The main idea is to embed the root lattice E_6 into E_8 and view the $U_{F(nX)}$ as certain commutant subalgebras of the lattice VOA $V_{\sqrt{2}E_8}$. Then we shall show that the product of two σ-involutions generated by $c = 6/7$ Virasoro vectors in $U_{F(nX)}$ exactly belong to the conjugacy class nX in Fi_{24}. By this procedure, we obtain a VOA description of the E_6 structure inside Fi_{24}.

The automorphism group of the Moonshine VOA $V^{\natural}$ is the Monster simple group $\mathbb{M}$ [FLM]. Consider the monstrous Griess algebra of dimension 196884 [C, G1]. It is known that the monstrous Griess algebra is naturally realized as the subspace of weight 2 of $V^{\natural}$ [FLM], which we call the Griess algebra of $V^{\natural}$ and denote by $\mathcal{G}^{\natural}$. We will freely use the character tables in [ATLAS].

5.1 The Fischer group vertex operator algebra $V\!F^{\natural}$

We denote by Fi_{24} the Fischer 3-transposition group and by Fi_{24}' its derived subgroup, the 3rd largest sporadic finite simple group. Let $g \in \mathbb{M}$ be a 3A-element. Then $C_{\mathbb{M}}(g) \simeq 3.\mathrm{Fi}_{24}'$ and it is shown in [C, MeN] that the monstrous Griess algebra $\mathcal{G}^{\natural}$ has an irreducible decomposition

$$\mathcal{G}^{\natural} = \underline{\mathbf{1}} \oplus \underline{\mathbf{1}} \oplus \underline{\mathbf{8671}} \oplus \underline{\mathbf{57477}} \oplus 2\cdot\underline{\mathbf{783}} \oplus 2\cdot\underline{\mathbf{64584}} \tag{5.1}$$

as a $C_{\mathbb{M}}(g)$-module. Therefore, we can take a Virasoro vector $u \in (\mathcal{G}^{\natural})^{C_{\mathbb{M}}(g)}$ such that $(\mathcal{G}^{\natural})^{C_{\mathbb{M}}(g)} = \mathbb{C}u \oplus \mathbb{C}(\omega - u)$ is an orthogonal sum. We take u to be the shorter one, that is, the central charge of u is smaller than that of $\omega - u$. It is also shown in loc. cit. that the central charge of the shorter Virasoro vector u is $4/5$ and its spectrum on $\mathcal{G}^{\natural}$ is as follows:

$$\begin{array}{ccccccccccccc} \mathcal{G}^{\natural} & = & \underline{\mathbf{1}} & \oplus & \underline{\mathbf{1}} & \oplus & \underline{\mathbf{8671}} & \oplus & \underline{\mathbf{57477}} & \oplus & 2\cdot\underline{\mathbf{783}} & \oplus & 2\cdot\underline{\mathbf{64584}} \\ u_{(0)} & : & 0 & & 2 & & 2/5 & & 0 & & 2/3 & & 1/15 \end{array} \tag{5.2}$$

The following result about the extendibility seems already known to experts (see for example [MeN, KMY, Mi2]), even though no rigorous proof has been given so far.

Theorem 5.1. *Let g be a 3A-element of the Monster. The cyclic group $\langle g\rangle$ uniquely determines an extendable simple $c = 4/5$ Virasoro vector in $(V^{\natural})^{C_{\mathbb{M}}(g)}$.*

Proof: By the decomposition in Eq. (5.1), every cyclic subgroup $\langle g\rangle$ defines a unique simple $c = 4/5$ Virasoro vector in $(V^\natural)^{C_{\mathbb{M}}(g)}$. We will prove that this Virasoro vector is extendable.

Let t be a 2A-involution in $N_{\mathbb{M}}(\langle g\rangle)$ ($\simeq 3.\mathrm{Fi}_{24}$) but not in $C_{\mathbb{M}}(g)$ ($\simeq 3.\mathrm{Fi}_{24}'$). Then the subgroup H generated by t and g in $\mathbb{M}$ is isomorphic to S_3 and $C_{\mathbb{M}}(H) \simeq \mathrm{Fi}_{23}$ (cf. [ATLAS] and Lemma 13.3 of [G1]). Take an involution $t' \in H$ such that $tt' = g$. Then t' is conjugate to t and H is generated by t and t'. By the one-to-one correspondence between 2A-elements of $\mathbb{M}$ and Ising vectors of $V^\natural$ (cf. [Mi1] and [Hö], Lemma 3; see also [HLY], Theorem. 5.1), there exist Ising vectors e^0, $e^1 \in V^\natural$ such that $\tau_{e^0} = t$, $\tau_{e^1} = t'$ and e^0 and e^1 are fixed by $C_{\mathbb{M}}(t)$ and $C_{\mathbb{M}}(t')$, respectively. Since $C_{\mathbb{M}}(H)$ is a subgroup of both $C_{\mathbb{M}}(t)$ and $C_{\mathbb{M}}(t')$, e^0 and e^1 are both contained in $(V^\natural)^{C_{\mathbb{M}}(H)}$. By [ATLAS], one obtains the following decomposition of the Griess algebra $\mathcal{G}^\natural$ as a $C_{\mathbb{M}}(H)$-module:

$$\mathcal{G}^\natural = 5\cdot\underline{\mathbf{1}} \oplus 3\cdot\underline{\mathbf{782}} \oplus 3\cdot\underline{\mathbf{3588}} \oplus \underline{\mathbf{5083}} \oplus \underline{\mathbf{25806}} \oplus \underline{\mathbf{30888}} \oplus 2\cdot\underline{\mathbf{60996}}. \tag{5.3}$$

Let U be the subalgebra of $V^\natural$ generated by e^0 and e^1. Then U is isomorphic to the 3A-algebra U_{3A} [Mi3, SY]. Since the Griess algebra of U is 4-dimensional (cf. Section 4.1), it follows from the decomposition above that the weight two subspace of $(V^\natural)^{C_{\mathbb{M}}(H)}$ is spanned by that of U and the conformal vector of $V^\natural$. Now it is clear that all simple $c = 4/5$ Virasoro vectors of $(V^\natural)^{C_{\mathbb{M}}(H)}$ are contained in U. The 3A-algebra has one extendable simple $c = 4/5$ Virasoro vector and three non-extendable ones. By Lemma 4.3, the non-extendable ones in the 3A-algebra have eigenvalues 13/8 on the Griess algebra of the 3A-subalgebra and these eigenvalues do not appear in the decomposition (5.2). Therefore, the subalgebra $(V^\natural)^{C_{\mathbb{M}}(g)}$ contains the unique simple $c = 4/5$ Virasoro vector, which is extendable as claimed. ∎

Let e^0 and e^1 be Ising vectors as in the proof above. Without loss, we may assume that the 3A-element ξ_u associated to u, defined as in Theorem 2.13, coincides with $\tau_{e^0}\tau_{e^1}$.

It follows from Theorem 2.13 (cf. [KMY, Mi2]) that for each embedding $\mathcal{W}(4/5) = L(4/5, 0) \oplus L(4/5, 3) \hookrightarrow V^\natural$, one can define an element ξ of the Monster with $\xi^3 = 1$. By computing its trace on $V_2^\natural$, one can show that ξ belongs to the conjugacy class 3A (cf. Section 4.1 of [Ma1]). We remark here that a single $L(4/5, 0)$ does not define an order three symmetry, and in order to obtain 3A-elements, we have to extend $L(4/5, 0)$ to a larger algebra $\mathcal{W}(4/5) = L(4/5, 0) \oplus L(4/5, 3)$ (cf. [Mi2]). It remains a natural question whether the map associating the order three element ξ defined as in Theorem 2.13 is injective or not. Since the order three element is defined not only by the Griess algebra but also by the 3-primary vector, we would need extra information about the weight three subspace of $V^\natural$ to solve this question.

Because of this problem, we first fix a 3A-element $g \in \mathbb{M}$ and then take the unique simple extendable $c = 4/5$ Virasoro vector u in the fixed point subalgebra $(V^\natural)^{C_{\mathbb{M}}(g)}$. Then $g = \xi_u$ by the preceding argument. Let w be an associated 3-primary vector of u and denote by $\mathcal{W}(u,w)$ the sub-VOA generated by u and w which is isomorphic to the extended Virasoro VOA $\mathcal{W}(4/5)$.

Definition 5.2. The *Fischer group vertex operator algebra* is defined as the commutant

$$VF^\natural := \mathrm{Com}_{V^\natural}(\mathcal{W}(u,w)) = \mathrm{Com}_{V^\natural}(\mathrm{Vir}(u)). \tag{5.4}$$

Below we will show that the VOA $VF^\natural$ affords an action of the Fischer 3-transposition group Fi_{24}.

Lemma 5.3. *$N_{\mathbb{M}}(\langle\xi_u\rangle) \subset \mathrm{Stab}_{\mathbb{M}}(u)$.*

Proof: Since $\mathrm{Aut}(\langle\xi_u\rangle) \simeq \mathrm{Aut}(\mathbb{Z}_3) = \mathbb{Z}_2$, we have either $N_{\mathbb{M}}(\langle\xi_u\rangle) = C_{\mathbb{M}}(\xi_u)$ or $C_{\mathbb{M}}(\xi_u)$ is normal of index 2 in $N_{\mathbb{M}}(\langle\xi_u\rangle)$. We have seen in Theorem 5.1 that $C_{\mathbb{M}}(\xi_u) \subset \mathrm{Stab}_{\mathbb{M}}(u)$. For elements $h \in C_{\mathbb{M}}(\xi_u)$ and $g \in N_{\mathbb{M}}(\langle\xi_u\rangle)$ we get $hgu = gh^g u = gu$ where $h^g = g^{-1}hg \in C_{\mathbb{M}}(\xi_u)$. Thus gu is fixed by $C_{\mathbb{M}}(\xi_u)$. By the decomposition (5.1) of $\mathcal{G}^\natural$ as a $C_{\mathbb{M}}(\xi_u)$-module we see that gu must be equal to the shorter Virasoro element u in the fixed point subspace $(\mathcal{G}^\natural)^{C_{\mathbb{M}}(\xi_u)}$. Thus $N_{\mathbb{M}}(\langle\xi_u\rangle) \subset \mathrm{Stab}_{\mathbb{M}}(u)$. ∎

Recall the group homomorphism ψ_u defined in (4.7). By Lemma 5.3, we have an action of $N_{\mathbb{M}}(\langle\xi_u\rangle)$ on $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(u))$ via ψ_u.

Proposition 5.4. *Consider the homomorphism $\psi_u : \mathrm{Stab}_{\mathbb{M}}(u) \to \mathrm{Aut}(\mathrm{Com}_{V^\natural}(\mathrm{Vir}(u)))$. Then $\psi_u(N_{\mathbb{M}}(\langle\xi_u\rangle)) \simeq \mathrm{Fi}_{24}$.*

Proof: It is clear that $\langle\xi_u\rangle \subset \ker\psi_u$. By (5.2), it is also clear that $\psi_u(C_{\mathbb{M}}(\xi_u)) \neq 1$. Therefore, either $\psi_u(N_{\mathbb{M}}(\langle\xi_u\rangle)) \simeq \mathrm{Fi}_{24}'$ or Fi_{24}. Since $\tau_{e^0} \in N_{\mathbb{M}}(\langle\xi_u\rangle) \setminus C_{\mathbb{M}}(\xi_u)$ acts non-trivially on $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(u))$ via ψ_u, we see $\psi_u(N_{\mathbb{M}}(\langle\xi_u\rangle)) \simeq \mathrm{Fi}_{24}$. ∎

By (5.1) and (5.2) the Griess algebra $VF_2^\natural$ is of dimension 57478 and has, under $\psi_u(N_{\mathbb{M}}(\langle\xi_u\rangle)) \simeq \mathrm{Fi}_{24}$, the decomposition

$$VF_2^\natural = \underline{\mathbf{1}} \oplus \underline{\mathbf{57477}}. \tag{5.5}$$

We have seen in Proposition 5.4 that $VF^\natural$ naturally affords an action of the full Fischer group Fi_{24}. We cannot prove that Fi_{24} is the full automorphism group of $VF^\natural$ at the moment. Instead, we will prove the following partial result.

Theorem 5.5. *Let $\mathcal{X}$ be the subalgebra of $VF^\natural$ generated by the weight 2 subspace. Then $\mathrm{Aut}(\mathcal{X}) \simeq N_{\mathbb{M}}(\langle\xi_u\rangle)/\langle\xi_u\rangle \simeq \mathrm{Fi}_{24}$.*

Proof: It is shown in [GL, LM] that there exists a 14-dimensional sublattice L of the Leech lattice Λ such that V_L^+ contains a sub-VOA isomorphic to U_{3A}. It is also known that the annihilator $\mathrm{Ann}_\Lambda(L) = \{\alpha \in \Lambda \mid \langle \alpha, \beta \rangle = 0 \text{ for all } \beta \in L\}$ contains a sublattice isomorphic to $\sqrt{2}A_2^{\oplus 5}$ (cf. [GL]). Therefore we can find a sub-VOA $U_{3A} \otimes V_{\sqrt{2}A_2}^+$ of V_Λ^+, which is also contained in the Moonshine VOA. This shows that $V\!F^\natural$ contains a sub-VOA isomorphic to $V_{\sqrt{2}A_2}^+$. It is well-known that $V_{\sqrt{2}A_2}^+$ contains 6 Ising vectors (cf. [LSY]) and therefore $V\!F^\natural$ contains at least 6 Ising vectors.

We have already seen that $\psi_u(N_{\mathbb{M}}(\langle \xi_u \rangle)) \simeq \mathrm{Fi}_{24}$ faithfully acts on $\mathcal{X}$. Let e^0 and e^1 be Ising vectors of $V^\natural$ such that $\tau_{e^0}\tau_{e^1} = \xi_u$. Then e^0 and e^1 generate a subalgebra $U(e^0, e^1)$ isomorphic to the 3A-algebra such that $\mathcal{W}(u, w) \subset U(e^0, e^1)$. We can take an Ising vector e^2 of $V^\natural$ orthogonal to both e^0 and e^1. Then $ge^2 \in \mathcal{X}$ and $\tau_{ge^2} \in C_{\mathbb{M}}(\xi_u)$ for any $g \in \mathrm{Aut}(\mathcal{X})$. As we have shown, $\ker \psi_u = \langle \xi_u \rangle$ and hence $\{\psi_u(\tau_{ge^2}) = g\psi_u(\tau_{e^2})g^{-1} \mid g \in \mathrm{Aut}(\mathcal{X})\}$ define non-trivial involutions on $\mathcal{X}$. Thus the subgroup generated by $\{\psi_u(\tau_{ge^2}) \mid g \in \mathrm{Aut}(\mathcal{X})\}$ is normal in $\mathrm{Aut}(\mathcal{X})$ and isomorphic to $C_{\mathbb{M}}(\xi_u)/\langle \xi_u \rangle \simeq \mathrm{Fi}_{24}'$.

Now by conjugation we can define a group homomorphism $\alpha : \mathrm{Aut}(\mathcal{X}) \to \mathrm{Aut}(\mathrm{Fi}_{24}') \simeq \mathrm{Aut}(\psi_u(C_{\mathbb{M}}(\xi_u)))$. Since $\mathrm{Out}(\mathrm{Fi}_{24}') = 2$ by [ATLAS] and $\psi_u(N_{\mathbb{M}}(\langle \xi_u \rangle))$ contains an outer involution defined by a simple $c = 6/7$ Virasoro vector $v \in U(e^0, e^1) \cap \mathcal{X}$, we see that

$$\alpha(\mathrm{Aut}(\mathcal{X})) = \alpha(\psi_u(N_{\mathbb{M}}(\langle \xi_u \rangle))) = \mathrm{Aut}(\mathrm{Fi}_{24}') \simeq \mathrm{Fi}_{24}.$$

This implies $\mathrm{Aut}(\mathcal{X}) \simeq \ker \alpha \times \mathrm{Fi}_{24}$. The Griess algebra of $V\!F^\natural$ is 57478-dimensional and it has a decomposition $57478 = \underline{\mathbf{1}} \oplus \underline{\mathbf{57477}}$ as a module over $\psi_u(C_{\mathbb{M}}(\xi_u)) \simeq \mathrm{Fi}_{24}'$. So the Griess algebra of $V\!F^\natural$ is spanned by its conformal vector $\omega_{V\!F^\natural}$ and Ising vectors $\{ge^2 \mid g \in C_{\mathbb{M}}(\xi_u)\}$. Now let $h \in \ker \alpha$. Then h acts by a scalar on the 57477-dimensional component, say λ. Write $e^2 = p\omega_{V\!F^\natural} + x$ with $p \in \mathbb{C}$ and $x \in \underline{\mathbf{57477}}$. Then $x \neq 0$ since the central charge of $\omega_{V\!F^\natural}$ is equal to $24 - 4/5$ and $he^2 = p\omega + \lambda x$. Since both $e^2/2$ and $he^2/2$ are idempotents in the Griess algebra, it follows that $\lambda = 1$ and $h = 1$. Therefore $\ker \alpha = 1$ and we obtain the desired isomorphism $\mathrm{Aut}(\mathcal{X}) = \psi_u(N_{\mathbb{M}}(\langle \xi_u \rangle)) \simeq \mathrm{Fi}_{24}$. ∎

5.2 The 3-transposition property

In this section, we will establish the correspondence between derived $c = 6/7$ Virasoro vectors in $V\!F^\natural$ and 2C-involutions of the Fischer group Fi_{24}.

Take a 2C-involution t of Fi_{24} and consider the decomposition of $V\!F^\natural$ as a module over $C_{\mathrm{Fi}_{24}}(t) \simeq \mathrm{Fi}_{23}$. In the computation of (5.3) we have already obtained that

$$V\!F_2^\natural = \underline{\mathbf{1}} \oplus \underline{\mathbf{1}} \oplus \underline{\mathbf{30888}} \oplus \underline{\mathbf{25806}} \oplus \underline{\mathbf{782}} \qquad (5.6)$$

as a $C_{\mathrm{Fi}_{24}}(t)$-module. Therefore, the fixed point subalgebra $(V\!F_2^\natural)^{C_{\mathrm{Fi}_{24}}(t)}$ is 2-dimensional and forms a commutative associative algebra spanned by two mutually orthogonal Virasoro vectors. In order to determine the central charge of the shorter Virasoro vector in $(VF^\natural)^{C_{\mathrm{Fi}_{24}}(t)}$, we use the 3A-algebra for the Monster to obtain suitable decompositions.

Let v be a derived $c = 6/7$ Virasoro vector in $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(u))$ with respect to u. Let $U \subset V^\natural$ be the corresponding sub-VOA isomorphic to U_{3A}, that is, $u+v$ is the conformal vector of U, and let e^0, e^1 and e^2 be Ising vectors of U such that $\tau_{e^0}\tau_{e^1} = \xi_u$.

For an irreducible U-module M, we set $H_M := \mathrm{Hom}_U(M, V^\natural)$. Then we have the decomposition

$$V^\natural = \bigoplus_{M \in \mathrm{Irr}(U_{3A})} M \otimes H_M. \tag{5.7}$$

Clearly H_M forms a module over the commutant subalgebra $\mathrm{Com}_{V^\natural}(U)$.

Lemma 5.6. *The top weight $h(H_M)$ and the dimension $d(H_M)$ of the top level of the $\mathrm{Com}_{V^\natural}(U)$-modules H_M are given by the following table*

M	$U(0)$	$U(1/7)$	$U(5/7)$	$U(2/5)$	$U(19/35)$	$U(4/35)$
$h(H_M)$	0	13/7	9/7	8/5	51/35	66/35
$d(H_M)$	1	25806	782	5083	3588	60996

(5.8)

Moreover, one has $\dim(H_{U(0)})_2 = 30889$.

Proof: By Lemma 2.6 and the classification of irreducible U_{3A}-modules in Theorem 4.4, we know that the possible eigenvalues of $u_{(1)}$ on $\mathcal{G}^\natural$ are 0, $1/15$, $2/5$ and $2/3$, and those for $v_{(1)}$ are 0, $1/7$, $5/7$, $4/3$, $1/21$ and $10/21$. Applying a similar computation as in the proof of Lemma 5.2 of [HLY], we obtain the Lemma. ∎

On $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(u))$, one can define the σ-involution σ_v as in (2.4), which coincides with $\psi_u(\tau_{e^0})$ by Lemma 4.8.

Proposition 5.7. *The involution $\sigma_v = \psi_u(\tau_{e^0})$ is a 2C-element of $\psi_u(N_{\mathbb{M}}(\langle \xi_u\rangle)) \simeq \mathrm{Fi}_{24}$.*

Proof: Let us consider the trace of $\sigma_v = \psi_u(\tau_{e^0})$ on the Griess algebra of $VF^\natural = \mathrm{Com}_{V^\natural}(\mathrm{Vir}(u))$. By (5.8) and (4.5), one has

$$\mathrm{Tr}_{VF^\natural_2}\sigma_v = 1 + \dim(H_{U(0)})_2 + \dim H_{U(5/7)} - \dim H_{U(1/7)} = 5866,$$

which coincides only with the trace of a 2C-involution of Fi_{24} on $VF^\natural_2 = \underline{\mathbf{1}} \oplus \underline{\mathbf{57477}}$ by [ATLAS]. ∎

Lemma 5.8. *The set $\{e^0, e^1, e^2\}$ consisting of the three Ising vectors of U is stabilized by $\psi_u^{-1}(C_{\mathrm{Fi}_{24}}(\psi_u(\tau_{e^0})))$.*

Proof: Take any $g \in \psi_u^{-1} C_{\mathrm{Fi}_{24}}(\psi_u(\tau_{e^0})) \subset N_{\mathbb{M}}(\langle \xi_u \rangle)$. Then $[g, \tau_{e^0}] \in \langle \xi_u \rangle$ and hence $\tau_{ge^0} = g\tau_{e^0}g^{-1} \in \{\tau_{e^0}, \xi_u\tau_{e^0}, \xi_u^2\tau_{e^0}\}$. Since $\tau_{e^0}\tau_{e^1} = \xi_u$ we get $\{\xi_u\tau_{e^0}, \xi_u^2\tau_{e^0}\} = \{\tau_{e^1}, \tau_{e^2}\}$. Thus, $ge^0 \in \{e^0, e^1, e^2\}$ by the one-to-one correspondence [Mi1, Hö]. Similarly, we also have $ge^1 \in \{e^0, e^1, e^2\}$ and $ge^2 \in \{e^0, e^1, e^2\}$. ∎

Proposition 5.9. *A derived $c = 6/7$ Virasoro vector $v \in \mathrm{Com}_{V^\natural}(\mathrm{Vir}(u))$ with respect to u is fixed by the centralizer of the 2C-involution $\psi_u(\tau_e)$ of the Fischer group* Fi_{24}.

Proof: Let $\alpha = e^0 + e^1 + e^2$. Then, α is fixed by $\psi_u^{-1} C_{\mathrm{Fi}_{24}}(\psi_u(\tau_{e^0}))$ by Lemma 5.8. On the other hand, by Lemma 4.2 and (4.2), we have

$$\alpha = e^0 + e^1 + e^2 = \frac{15}{32}u + \frac{21}{16}v \quad \text{and} \quad \alpha^2 = 2\left(\frac{15}{32}\right)^2 u + 2\left(\frac{21}{16}\right)^2 v,$$

where v is the derived $c = 6/7$ Virasoro vector in U. Thus,

$$v = \frac{16}{567}(16\alpha^2 - 15\alpha).$$

Hence, v is fixed by the centralizer of the 2C-involution $\psi_u(\tau_{e^0})$ in Fi_{24}. ∎

Now we establish the one-to-one correspondence between 2C-involutions of Fi_{24} and derived $c = 6/7$ Virasoro vectors of $V\!F^\natural$, which is one of our main results.

Theorem 5.10. *The map which associates a derived $c = 6/7$ Virasoro vector to its σ-involution defines a bijection between the set of all derived $c = 6/7$ Virasoro vectors in $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(u))$ with respect to u and the 2C-conjugacy class of $\mathrm{Fi}_{24} = \psi_u(N_{\mathbb{M}}(\langle \xi_u \rangle))$.*

Proof: The map of the theorem is equivariant with respect to the natural action of $\psi_u(N_{\mathbb{M}}(\langle \xi_u \rangle))$ on the derived vectors and the conjugation action of Fi_{24} on the set of its 2C-involutions, respectively.

As seen in the proof of Theorem 5.1, the vector u is contained in a sub-VOA isomorphic to the 3A-algebra U_{3A}. Thus there exists at least one derived $c = 6/7$ Virasoro vector with respect to u in $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(u))$. The transitivity of the conjugation action on the 2C-involutions shows now the surjectivity of the map.

For the injectivity, fix a 2C-involution t of $\mathrm{Fi}_{24} = \psi_u(N_{\mathbb{M}}(\langle \xi_u \rangle))$. By Proposition 5.9, any derived $c = 6/7$ Virasoro vector v in $\mathrm{Com}_{V^\natural}(\mathrm{Vir}(u))$ such that $\sigma_v = t$ is contained in $(V\!F^\natural)^{C_{\mathrm{Fi}_{24}}(t)}$. We have seen in (5.6) that the fixed point subalgebra $(V\!F_2^\natural)^{C_{\mathrm{Fi}_{24}}(t)}$ is spanned by two mutually orthogonal Virasoro vectors. Hence v must be the unique shorter Virasoro vector of $(V\!F^\natural)^{C_{\mathrm{Fi}_{24}}(t)}$ of central charge $c = 6/7$. ∎

The proof gives also the following corollary.

Corollary 5.11. *Every 2C-involution t of the Fischer group Fi_{24} defines an unique derived $c = 6/7$ Virasoro vector of the fixed point subalgebra $(V\!F^\natural)^{C_{\mathrm{Fi}_{24}}(t)}$.*

As a consequence of Theorem 4.10 and Theorem 5.10, we also recover:

Corollary 5.12. *The* $2C$ *involutions of the Fischer group* Fi_{24} *satisfy the* 3*-transposition property.*

Remark 5.13. We expect that the full automorphism group of $V\!F^\natural$ is actually Fi_{24}. Because of Theorem 5.5, it suffices to show $V\!F^\natural = \mathcal{X}$, that is, $V\!F^\natural$ is generated by its Griess algebra. This is a technically difficult problem since we do not know a nice embedding of $\mathcal{W}(4/5)$ into $V^\natural$ to study the commutant subalgebra $V\!F^\natural = \mathrm{Com}_{V^\natural}(\mathcal{W}(4/5))$. It is conjectured in [DM] that one can obtain $V^\natural$ from V_Λ by the $\mathbb{Z}_3$-orbifold construction where the $\mathbb{Z}_3$-automorphism is induced by a automorphism of the A_2 lattice via an embedding $\sqrt{2}A_2^{12} \hookrightarrow \Lambda$. If this conjectural $\mathbb{Z}_3$-orbifold construction is established, we obtain a natural embedding of $\mathcal{W}(4/5)$ into $V^\natural$ and then we can solve the question above immediately by a decomposition of $V^\natural$ given in [KLY].

5.3 Embedding of $U_{F(nX)}$ into $V\!F^\natural$

We recall the definition of $V_{F(nX)}$ from Section 3.1. We have a full rank sublattice $Q \oplus E_6 \simeq A_2 \oplus E_6$ of E_8. Since the index of $A_2 \oplus E_6$ in E_8 is three, we have a coset decomposition

$$E_8 = A_2 \oplus E_6 \sqcup (\delta + A_2 \oplus E_6) \sqcup (2\delta + A_2 \oplus E_6)$$

with some $\delta \in E_8$ and correspondingly we obtain a decomposition

$$V_{\sqrt{2}E_8} = V_{\sqrt{2}(A_2 \oplus E_6)} \oplus V_{\sqrt{2}(\delta + A_2 \oplus E_6)} \oplus V_{\sqrt{2}(2\delta + A_2 \oplus E_6)}.$$

Define $\eta \in \mathrm{Aut}(V_{\sqrt{2}E_8})$ by

$$\eta = \begin{cases} 1 & \text{on} \quad V_{\sqrt{2}(A_2 \oplus E_6)}, \\ e^{2\pi\sqrt{-1}/3} & \text{on} \quad V_{\sqrt{2}(\delta + A_2 \oplus E_6)}, \\ e^{4\pi\sqrt{-1}/3} & \text{on} \quad V_{\sqrt{2}(2\delta + A_2 \oplus E_6)}. \end{cases}$$

Then η is clearly in $\mathcal{F}_{nX}$, see (3.10). Indeed, $\mathcal{F}_{nX}$ is generated by η and ρ_{nX}. Note that we can write down ρ_{nX} in exponential form

$$\rho_{nX} = \exp(2\pi\sqrt{-1}\,\gamma^{nX}_{(0)}/n) \quad \text{with suitable } \gamma^{nX} \in L^*_{nX},$$

which also defines an automorphism of $V_{\sqrt{2}E_8}$ and fixes $V_{\sqrt{2}\tilde{L}_{nX}}$ pointwisely.

Remark 5.14. Recall $\tilde{\omega}_Q$ with $Q \simeq A_2$ is a simple extendable $c = 4/5$ Virasoro vector in a lattice VOA $V_{\sqrt{2}Q}$ and $U_{F(nX)}$ equals the commutant subalgebra $\mathrm{Com}_{V_{F(nX)}}(\mathrm{Vir}(\tilde{\omega}_Q))$ in $V_{F(nX)}$. Moreover, ρ_{nX} fixes $\tilde{\omega}_Q$. Let U^1 be the subalgebra generated by $\hat{e}$ and $\eta(\hat{e})$, and set $U^2 = \rho_{nX}(U^1)$. Then $U^1 \simeq U^2 \simeq U_{3A}$ and $\tilde{\omega}_Q$ is contained in both U^1 and U^2. Note that $\tilde{v} = \tilde{\omega}_{E_6}$ is contained in $\mathrm{Com}_{U^1}(\mathrm{Vir}(\tilde{\omega}_Q))$. Similarly, $\tilde{v}' = \rho_{nX}\tilde{v}$ of $U_{F(nX)}$ is contained in $\mathrm{Com}_{U^2}(\rho_{nX}(\mathrm{Vir}(\tilde{\omega}_Q))) = \mathrm{Com}_{U^2}(\mathrm{Vir}(\tilde{\omega}_Q))$. Thus, the $c = 6/7$ Virasoro vectors $\tilde{v}$ and $\tilde{v}'$ of $U_{F(nX)}$ are derived Virasoro vectors with respect to $\tilde{\omega}_Q$.

Proposition 5.15. *For any $nX = 1A$, $2A$ or $3A$, the VOA $V_{F(nX)}$ can be embedded into the Moonshine VOA $V^\natural$.*

Proof: As we have shown in Section 3.2, $V_{F(1A)}$ is isomorphic to the monstrous 3A-algebra U_{3A} and $V_{F(2A)}$ is isomorphic to the monstrous 6A-algebra U_{6A} discussed in [LYY2]. It is shown in [LM] that both U_{3A} and U_{6A} are subalgebras of $V^\natural$ and therefore $V_{F(1A)}$ and $V_{F(2A)}$ are also contained in $V^\natural$. That $V_{F(3A)} \simeq M_{\mathcal{C}_4}$ is contained in $V^\natural$ will be shown in Appendix A. ∎

Finally, we will establish our main theorem.

Theorem 5.16. *Let u be a simple extendable $c = 4/5$ Virasoro vector in $V^\natural$ such that $u \in (V^\natural)^{C_{\mathbb{M}}(\xi_u)}$. Then for any $nX = 1A$, $2A$, $3A$, the VOA $U_{F(nX)}$ can be embedded into $V\!F^\natural = \mathrm{Com}_{V^\natural}(\mathrm{Vir}(u))$. Moreover, $\sigma_{\tilde{v}}\sigma_{\tilde{v}'}$ belongs to the conjugacy class nX of* $\mathrm{Fi}_{24} \subset \mathrm{Aut}(V\!F^\natural)$.

Proof: First we embed $V_{F(nX)}$ into $V^\natural$ using Proposition 5.15. Let e, e' be a pair of Ising vectors of $V_{F(nX)}$ that generate a subalgebra U such that $\tilde{\omega}_Q \in U$ and $U \simeq U_{3A}$. Since pairs of Ising vectors in $V^\natural$ generating the 3A-algebra are mutually conjugate under $\mathrm{Aut}(V^\natural)$, we may identify $\tilde{\omega}_Q$ with u by Theorem 5.1. Thus, we have

$$U_{F(nX)} = \mathrm{Com}_{V_{F(nX)}}(\mathrm{Vir}(\tilde{\omega}_{Q(E_6)})) \subset \mathrm{Com}_{V^\natural}(\mathrm{Vir}(u)) = V\!F^\natural$$

as desired.

Next we will show that $h := \sigma_{\tilde{v}}\sigma_{\tilde{v}'}$ belongs to the class nX of $\mathrm{Fi}_{24} = \mathrm{Aut}(\mathfrak{X})$. Note that $\tilde{v}$, $\tilde{v}' \in V\!F^\natural_2 \subset \mathfrak{X}$. Recall that there is an exact sequence

$$1 \longrightarrow \langle \xi_u \rangle \longrightarrow N_{\mathbb{M}}(\langle \xi_u \rangle) \longrightarrow \mathrm{Aut}(\mathfrak{X}) \simeq \mathrm{Fi}_{24} \longrightarrow 1$$

with the projection map $\psi_u : N_{\mathbb{M}}(\langle \xi_u \rangle) \to \mathrm{Aut}(\mathfrak{X})$. Let e^1 and e^2 be Ising vectors in V_{nX} such that $\psi_u(\tau_{e^1}) = \sigma_{\tilde{v}}$ and $\psi_u(\tau_{e^2}) = \sigma_{\tilde{v}'}$. Set $g = \tau_{e^1}\tau_{e^2}$. Then $h = \psi_u(g)$ and the inverse image $\psi_u^{-1}(\langle h \rangle)$ has order $3n$ and is generated by ξ_u and g.

1A case: In this case, $\tilde{v} = \tilde{v}'$ and hence $h = \sigma_{\tilde{v}}\sigma_{\tilde{v}'}$ belongs to the class 1A.

2A case: In this case, $V_{F(2A)} \simeq U_{6A}$. Then $g = \tau_{e^1}\tau_{e^2}$ has order 2 or 6 and the group generated by ξ_u and g is a cyclic group of order 6 which is generated by a 6A-element of $\mathbb{M}$. Let t be the unique involution in $\langle \xi_u, g \rangle$. Then by [ATLAS], t belongs to the 2A conjugacy class of $\mathbb{M}$ and $C_{\mathbb{M}}(t)$ is isomorphic to a double cover of the Baby Monster $\mathbb{B}$. Thus we have an exact sequence

$$1 \longrightarrow \langle t \rangle \longrightarrow C_{\mathbb{M}}(t) \xrightarrow{\varphi_t} \mathbb{B} \longrightarrow 1.$$

Since $\langle t \rangle$ and $\langle \xi_u \rangle$ are unique order 2 and order 3 subgroups in $\langle \xi_u, g \rangle$ and t commutes with ξ_u, we have

$$N_{\mathbb{M}}(\langle \xi_u, g \rangle) = N_{\mathbb{M}}(\langle \xi_u, t \rangle) = C_{\mathbb{M}}(t) \cap N_{\mathbb{M}}(\langle \xi_u \rangle).$$

Set $G = N_{\mathbb{M}}(\langle \xi_u, g \rangle)$. Then

$$\varphi_t(G) = N_{\mathbb{B}}(\langle \varphi_t(\xi_u) \rangle) \quad \text{and} \quad \psi_u(G) = C_{\mathrm{Fi}_{24}}(\psi_u(t)).$$

Note that $\varphi_t(\xi_u)$ has order 3 and $\psi_u(g) = \psi_u(t)$ has order 2 since $(2, 3) = 1$. By comparing the 3-local subgroups of $\mathbb{B}$ and the 2-local subgroups of Fi_{24} in [ATLAS], we have

$$\varphi_t(G) \simeq \mathrm{S}_3 \times \mathrm{Fi}_{22}{:}2 \quad \text{and} \quad \psi_u(G) = C_{\mathrm{Fi}_{24}}(\psi_u(g)) \simeq (2 \times 2.\mathrm{Fi}_{22}){:}2.$$

Thus, $h = \psi_u(g)$ belongs to the conjugacy class 2A of Fi_{24} by [ATLAS].

3A case: In this case, $V_{F(3A)} \simeq M_{\mathcal{C}_4}$, the ternary code VOA associated to the tetra code $\mathcal{C}_4$ and ξ_u, τ_{e^1} and τ_{e^2} generate a subgroup of the shape $3^2{:}2$, which has exactly 4 distinct subgroups of order 3. Since $V_{F(3A)} \supset \mathcal{W}(4/5)^{\otimes 4}$ and each $\mathcal{W}(4/5)$ defines a non-trivial subgroup of order 3, all order 3 elements of $\langle \xi_u, \tau_{e^1}, \tau_{e^2} \rangle$ belong to the conjugacy class 3A of $\mathbb{M}$.

Let $g = \tau_{e^1}\tau_{e^2}$. Then ξ_u and g generate a 3A-pure elementary abelian 3-subgroup of order 3^2 in $\mathbb{M}$. The normalizer $N_{\mathbb{M}}(\langle \xi_u, g \rangle)$ has the shape $(3^2 : 2 \times \mathrm{O}_8^+(3)).\mathrm{S}_4$ while the centralizer $C_{\mathbb{M}}(\langle \xi_u, g \rangle)$ has the shape $3^2 \times \mathrm{O}_8^+(3)$ (cf. [Wi] and page 234 of [ATLAS]). Thus, $N_{\mathbb{M}}(\langle \xi_u, g \rangle)$ acts (by conjugation) on $\langle \xi_u, g \rangle$ as $2\mathrm{S}_4 (\simeq \mathrm{GL}_2(3))$ and S_4 acts as permutations of the 4 distinct subgroups of order 3 in $\langle \xi_u, g \rangle$. Thus, $N_{\mathbb{M}}(\langle \xi_u \rangle) \cap N_{\mathbb{M}}(\langle \xi_u, g \rangle)$ has the shape $(3^2{:}2 \times \mathrm{O}_8^+(3)).\mathrm{S}_3$ and $\psi_u(N_{\mathbb{M}}(\langle \xi_u \rangle) \cap N_{\mathbb{M}}(\langle \xi_u, g \rangle))$ has the shape $\mathrm{S}_3 \times \mathrm{O}_8^+(3).\mathrm{S}_3$. Since $\psi_u(\langle \xi_u, g \rangle) = \psi_u(\langle g \rangle)$, we see that $\psi_u(N_{\mathbb{M}}(\langle \xi_u \rangle) \cap N_{\mathbb{M}}(\langle \xi_u, g \rangle))$ normalizes the subgroup $\langle \psi_u(g) \rangle$ and hence $\psi_u(N_{\mathbb{M}}(\langle \xi_u \rangle) \cap N_{\mathbb{M}}(\langle \xi_u, g \rangle)) < N_{\mathrm{Fi}_{24}}(\langle \psi_u(g) \rangle)$.

By [ATLAS], page 207, $\psi_u(N_{\mathbb{M}}(\langle \xi_u \rangle) \cap N_{\mathbb{M}}(\langle \xi_u, g \rangle))$ is isomorphic to $N_{\mathrm{Fi}_{24}}(3A)$, where $N_{\mathrm{Fi}_{24}}(3A)$ denotes the normalizer of a cyclic subgroup generated by a 3A-element in Fi_{24}, and it is a maximal subgroup of Fi_{24}. Thus, $N_{\mathrm{Fi}_{24}}(\langle \psi_u(g) \rangle) \simeq N_{\mathrm{Fi}_{24}}(3A)$ and $h = \psi_u(g)$ belongs to the conjugacy class 3A of Fi_{24}. ∎

Remark 5.17. We note that in the 3A case, the group $N_{\mathbb{M}}(\langle \xi_u, g \rangle)$ acts on $V_{F(3A)} \simeq M_{\mathcal{C}_4}$ as $(3^2 : 2)\mathrm{S}_4$, which is isomorphic to $\mathrm{Aut}(M_{\mathcal{C}_4})$ (cf. Remark 3.7). In fact, by the similar argument as in Remark 3.9, one can show without using the property of the Monster that the whole group $\mathrm{Aut}(M_{\mathcal{C}_4})$ can be extended to a subgroup of $\mathrm{Aut}(V^\natural)$ since all irreducible modules of $M_{\mathcal{C}_4}$ can be embedded into $V_{\sqrt{2}E_8}$ and are $\mathrm{Aut}(M_{\mathcal{C}_4})$-invariant.

A Appendix: Embedding of $V_{F(3A)}$ into $V^\natural$

In this Appendix, we give an embedding of $V_{F(3A)}$ into the moonshine VOA which completes the proof of Proposition 5.15. We achieve this by providing an explicit embedding of $V_{F(3A)} \simeq M_{\mathcal{C}_4}$ into $V_\Lambda^+ \subset V^\natural$, where $M_{\mathcal{C}_4}$ refers to the ternary code VOA associated to the tetra code $\mathcal{C}_4$ constructed in [KMY] (see Eq. (3.13)). The main idea is essentially given in [GL2] and [GL3] (see also [LM]).

First, we consider some automorphisms of V_{A_2} and V_{E_8}. Set $a_i = E_{i,i} - E_{i+1,i+1}$, $x_i^+ = E_{i,i+1}$ and $x_i^- = E_{i+1,i}$ for $i = 1, 2$, where $E_{i,j}$ denotes a 3×3 matrix whose (i,j)-entry is 1 and others are 0. Then $\{a_i, x_i^\pm \mid i = 1, 2\}$ is a set of Chevalley generators of $\mathfrak{sl}_3(\mathbb{C})$. Let α_1, α_2 be simple roots of the root lattice A_2. The weight one subspace of $(V_{A_2})_1$ of the lattice VOA $V_{A_2} \simeq M_{\mathbb{C}A_2}(1) \otimes \mathbb{C}[A_2]$ forms a Lie algebra isomorphic to $\mathfrak{sl}_3(\mathbb{C})$ by the following correspondence:

$$a_i \longmapsto \alpha_{i(-1)}\mathbb{1}, \quad x_i^\pm \longmapsto \pm e^{\pm\alpha_i}. \tag{A.1}$$

The automorphism group of the lattice VOA V_{A_2} is isomorphic to the automorphism group of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$, which is isomorphic to $\mathrm{PSL}_3(\mathbb{C}) \rtimes \mathbb{Z}_2$. Since V_{A_2} is generated by its weight one subspace, $\mathrm{SL}_3(\mathbb{C})$ acts on V_{A_2} via the adjoint map [FLM] under the identification above. Namely, $P \in \mathrm{SL}_3(\mathbb{C})$ acts by $A \mapsto A^P = P^{-1}AP$ for $A \in \mathfrak{sl}_3(\mathbb{C}) \simeq (V_{A_2})_1$.

Let $\zeta = e^{2\pi i/3}$ be a cubic root of unity and consider the elements

$$\tau := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad s := \frac{1}{\sqrt{3}} \begin{pmatrix} \zeta & \zeta^2 & 1 \\ \zeta^2 & \zeta & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{A.2}$$

in $\mathrm{SU}_3 \subset \mathrm{SL}_3(\mathbb{C})$. Then

$$r := s^{-1}\tau s = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.3}$$

Let A_2^* be the dual lattice of A_2. We can describe the action of r and τ on V_{A_2} and the V_{A_2}-module $V_{A_2^*}$ using this identification more explicitly. Let $\delta = \alpha_1 + \alpha_2$ be the half sum of positive roots, i.e., the Weyl vector. Then $r(u \otimes e^\alpha) = \zeta^{\langle \delta, \alpha \rangle} u \otimes e^\alpha$ for all $u \in M(1)$ and $\alpha \in A_2^*$. Since τ normalizes the canonical Cartan subalgebra of $\mathfrak{sl}_3(\mathbb{C})$, it induces an order 3 automorphism $\bar{\tau}$ of the root lattice A_2 explicitly given by

$$\bar{\tau}: \quad \alpha_1 \longmapsto \alpha_2 \longmapsto -(\alpha_1 + \alpha_2) \longmapsto \alpha_1.$$

Now fix a sublattice of type $A_2^4 = A_2 \perp A_2 \perp A_2 \perp A_2$ in E_8. Then $E_8/A_2^4 \subset (A_2^*/A_2)^4 \simeq (\mathbb{Z}/3\mathbb{Z})^4$ can be identified with the tetra code $\mathcal{C}_4$. The corresponding inclusion $V_{A_2}^{\otimes 4} \subset V_{E_8}$ induces an action of $\mathrm{SL}_3(\mathbb{C})^4$ on V_{E_8} by automorphisms where the center $(\mathbb{Z}/3\mathbb{Z})^4$ of $\mathrm{SL}_3(\mathbb{C})^4$ acts via its quotient $\widehat{\mathcal{C}_4}$, the dual group of $\mathcal{C}_4$.

Define

$$
\begin{array}{rclrcl}
\tilde{h}_1 & = & 1 \otimes \tau \otimes \tau \otimes \tau, & \qquad \tilde{h}_2 & = & \tau \otimes \tau \otimes \tau^{-1} \otimes 1, \\
\rho_1 & = & 1 \otimes r \otimes r \otimes r, & \qquad \rho_2 & = & r \otimes r \otimes r^{-1} \otimes 1, \\
\tilde{s} & = & s \otimes s \otimes s \otimes s & & &
\end{array}
$$

as automorphisms of V_{E_8}.

Let $\mathcal{G}$ be the subgroup generated by $\tilde{h}_1$ and $\tilde{h}_2$ and $\mathcal{F}$ be the subgroup generated by ρ_1 and ρ_2. Then $\mathcal{F} \simeq \mathcal{G} \simeq 3^2$. Moreover, by (A.3), we have

$$
\mathcal{F} = \tilde{s}^{-1} \mathcal{G} \tilde{s}. \tag{A.4}
$$

Remark A.1. We shall note that $\{x \in E_8 \mid \langle x, (0, \delta, \delta, \delta)\rangle \in 3\mathbb{Z}\} \simeq E_6 \perp A_2$ and $\{x \in E_8 \mid \langle x, (\delta, \delta, -\delta, 0)\rangle \in 3\mathbb{Z}\} \simeq E_6 \perp A_2$. Moreover, we have

$$
K := \{x \in E_8 \mid \langle x, (0, \delta, \delta, \delta)\rangle \in 3\mathbb{Z} \text{ and } \langle x, (\delta, \delta, -\delta, 0)\rangle \in 3\mathbb{Z}\} \simeq A_2^4. \tag{A.5}
$$

Thus, we have $V_{E_8}^{\langle \rho_1 \rangle} \simeq V_{E_8}^{\langle \rho_2 \rangle} \simeq V_{E_6} \otimes V_{A_2}$ and $V_{E_8}^{\langle \rho_1, \rho_2 \rangle} \simeq V_{A_2^4}$. However, we shall remark that the sublattice $K \simeq A_2^4$ obtained in (A.5) is not the same A_2^4 used to define h_i and ρ_i, $i = 1, 2$.

Set

$$
h_1 = \mathrm{id} \oplus \bar{\tau} \oplus \bar{\tau} \oplus \bar{\tau} \qquad \text{and} \qquad h_2 = \bar{\tau} \oplus \bar{\tau} \oplus \bar{\tau}^{-1} \oplus \mathrm{id}
$$

Then h_1 and h_2 can be considered as isometries of $E_8 \subset (A_2^*)^4$ induced by $\tilde{h}_1$ and $\tilde{h}_2$ and they also generate a subgroup isomorphic to 3^2.

Now consider the following sublattices of an orthogonal sum $E_8 \perp E_8$. Let

$$
\begin{aligned}
R &= \{(x, x) \in E_8 \perp E_8 \mid x \in E_8\}, \\
R^1 &= \{(x, h_1 x) \in E_8 \perp E_8 \mid x \in E_8\}, \\
R^2 &= \{(x, h_2 x) \in E_8 \perp E_8 \mid x \in E_8\}.
\end{aligned}
$$

Then $R \simeq R^1 \simeq R^2 \simeq \sqrt{2} E_8$. Note also that $R^1 = (\mathrm{id} \oplus h_1) R$ and $R^2 = (\mathrm{id} \oplus h_2) R$.

Let

$$
e_R = \frac{1}{16} \omega_R + \frac{1}{32} \sum_{\alpha \in R_4} e^{\alpha}
$$

be the Ising vector associated to R defined by (3.11), where $R_4 = \{\alpha \in R \mid \langle \alpha, \alpha \rangle = 4\}$.

Now let $\widetilde{\mathcal{F}} = \{\mathrm{id} \otimes \rho \mid \rho \in \mathcal{F}\}$. Then $\widetilde{\mathcal{F}}$ stabilizes the lattice VOA V_R and by Remark A.1, the fixed point sub-VOA $V_R^{\widetilde{\mathcal{F}}} \simeq V_{\sqrt{2}A_2^4}$. By the definition of $V_{F(3A)}$ (see the 3A case in Sec. 3.2), we can obtain a sub-VOA isomorphic to $V_{F(3A)}$ in $V_R \simeq V_{\sqrt{2}E_8}$. By Lemma 3.8, this $V_{F(3A)}$ is generated by e_R, $(\mathrm{id} \otimes \rho_1)e_R$ and $(\mathrm{id} \otimes \rho_2)e_R$.

Consider the automorphism $\hat{\sigma} := \tilde{s} \otimes \tilde{s}^{-1}$ of $V_{E_8 \perp E_8} \simeq V_{E_8} \otimes V_{E_8}$. The following lemma is essentially proved in [GL3] with some trivial modification.

Lemma A.2 (Lemma 2.19 of [GL3])**.** *We have $\hat{\sigma}^{-1}e_R \in V_R$.*

Next, we will show that $\hat{\sigma}^{-1}e$ is in fact in V_R^+. For any even lattice L, let $\theta : V_L \to V_L$ be the involution defined by

$$\theta : \alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes e^{\alpha} \longmapsto (-1)^k \alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes e^{-\alpha} \tag{A.6}$$

(cf. [FLM, Mi1]). Note that if L is a root lattice of type A_2, by identifying $(V_{A_2})_1$ with $\mathfrak{sl}_3(\mathbb{C})$ as in (A.1) we have

$$A^{\theta} = -{}^tA, \qquad A \in \mathfrak{sl}_3(\mathbb{C}).$$

If we extend θ to a map on $V_{A_2^*}$ then we have

$$\mathrm{Aut}(\mathrm{SL}_3(\mathbb{C})) \simeq \mathrm{SL}_3(\mathbb{C}) \rtimes \langle \theta \rangle$$

where θ acts by $X \mapsto {}^tX^{-1}$ on $\mathrm{SL}_3(\mathbb{C})$.

Lemma A.3. *The Ising vectors $\hat{\sigma}^{-1}e_R$, $(\mathrm{id} \otimes \tilde{h}_1)\hat{\sigma}^{-1}e_R$ and $(\mathrm{id} \otimes \tilde{h}_2)\hat{\sigma}^{-1}e_R$ are fixed by θ.*

Proof: Let s be defined as in (A.2). Then we have ${}^ts = s$ and $s^4 = 1$. Thus s and θ generate a dihedral group of order 8. One has $\theta s \theta s^{-1} = \theta s^{-1} \theta s = s^2$ as a direct calculation shows.

Now we consider the action of $\tilde{s}$ and θ on V_{E_8}. This action is given by the diagonal embedding of $\langle s, \theta \rangle$ into $(\mathrm{Aut}(\mathrm{SL}_3(\mathbb{C}))^4$. We get $\theta \tilde{s} \theta \tilde{s}^{-1} = \theta \tilde{s}^{-1} \theta \tilde{s} = \tilde{s}^2$.

Since s^2 is a permutation matrix, s^2 normalizes the Cartan subalgebra $\mathbb{C}a_1 + \mathbb{C}a_2$ of $\mathfrak{sl}_3(\mathbb{C})$. Moreover, $\tilde{s}^2$ normalizes the standard Cartan subalgebra of $(V_{E_8})_1$. That means

$$\tilde{s}^2(M_{\mathbb{C}E_8}(1)) = M_{\mathbb{C}E_8}(1).$$

Thus $\tilde{s}^2$ induce an isometry $\mu := \overline{\tilde{s}^2}$ of the root lattice E_8 such that

$$\tilde{s}^2(e^{\alpha}) = \epsilon(\alpha) e^{\mu\alpha} \quad \text{for } \alpha \in E_8, \tag{A.7}$$

where $\epsilon(\alpha) = \pm 1$.

Since $e^{(\alpha,\alpha)} = e^{(\alpha,0)}{}_{(-1)}e^{(0,\alpha)}$ in $V_{E_8\oplus E_8} \simeq V_{E_8}\otimes V_{E_8}$, one has

$$\theta\hat{\sigma}\theta\hat{\sigma}^{-1}(e^{(\alpha,\alpha)}) = \theta\hat{\sigma}\theta\hat{\sigma}^{-1}(e^{(\alpha,0)}{}_{(-1)}e^{(0,\alpha)}) = (\theta\tilde{s}\theta\tilde{s}^{-1}(e^{(\alpha,0)}))_{(-1)}(\theta\tilde{s}^{-1}\theta\tilde{s}(e^{(0,\alpha)}))$$

$$= (\epsilon(\alpha)e^{(\mu\alpha,0)})_{(-1)}(\epsilon(\alpha)e^{(0,\mu\alpha)}) = e^{(\mu\alpha,\mu\alpha)}$$

for any $\alpha \in E_8$. Therefore, $\theta\hat{\sigma}\theta\hat{\sigma}^{-1}$ fixes e_R and so does $\hat{\sigma}\theta\hat{\sigma}^{-1} = \theta(\theta\hat{\sigma}\theta\hat{\sigma}^{-1})$ since θ also fixes e_R. Thus, $\hat{\sigma}^{-1}e_R$ is fixed by θ. Since $\mathrm{id}\otimes\tilde{h}_1$ and $\mathrm{id}\otimes\tilde{h}_2$ commute with θ, $(\mathrm{id}\otimes\tilde{h}_1)\hat{\sigma}^{-1}e_R$ and $(\mathrm{id}\otimes\tilde{h}_2)\hat{\sigma}^{-1}e_R$ are also fixed by θ. ∎

Using (A.4) we have that $\hat{\sigma}^{-1}(V_{F(3A)})$ is generated by $\{(\mathrm{id}\otimes\tilde{h})\hat{\sigma}^{-1}e_R \mid \tilde{h}\in\mathcal{G}\}$ or $\{\hat{\sigma}^{-1}e_R, (\mathrm{id}\otimes\tilde{h}_1)\hat{\sigma}^{-1}e_R, (\mathrm{id}\otimes\tilde{h}_2)\hat{\sigma}^{-1}e_R\}$.

Since $(\mathrm{id}\oplus h_1)R = R^1$ and $(\mathrm{id}\oplus h_2)R = R^2$, we have $(\mathrm{id}\otimes\tilde{h}_1)\hat{\sigma}^{-1}e_R \in V_{R^1}^+$ and $(\mathrm{id}\otimes\tilde{h}_2)\hat{\sigma}^{-1}e_R \in V_{R^2}^+$, by Lemma A.3. Hence, we have $\hat{\sigma}^{-1}(V_{F(3A)}) < V_{R+R^1+R^2}^+$. Therefore, it remains to show that $L = R + R^1 + R^2$ can be embedded into the Leech lattice Λ.

First, we recall the ternary Golay code construction of the Leech lattice Λ [CS].

Let S be an orthogonal sum of 12 copies of A_2. Then the discriminant group S^*/S has a natural identification with $\mathbb{Z}_3^{12}$. The ternary Golay code $\mathcal{C}_{12}\subset\mathbb{Z}_3^{12}$ can be defined using the tetracode $\mathcal{C}_4$ as the set

$$\mathcal{C}_{12} = \{(c^0, -c^+, c^-) \mid c^0, c^+, c^- \in \mathbb{Z}_3^4,\ c^0 + c^+ + c^- = -\textstyle\sum_{i=1}^4 c_i^0\cdot(1,1,1,1),\ c^+ - c^- \in \mathcal{C}_4\}.$$

For each codeword $x = (x_1,\dots,x_{12}) \in \mathcal{C}_{12}$, let $\gamma_x = (\gamma_{x_1},\dots,\gamma_{x_{12}}) \in S^*$ be some vector which modulo S gives the codeword x. Then

$$\mathcal{N} := \bigcup_{x\in\mathcal{C}_{12}} (\gamma_x + S)$$

is isometric to the Niemeier lattice of type A_2^{12}.

Let $\delta = \alpha_1 + \alpha_2$ be the half sum of positive roots of A_2 and let

$$\hat{\delta} := (\delta,\delta,\delta,\delta,-\delta,-\delta,-\delta,-\delta,\delta,\delta,\delta,\delta).$$

Then

$$\mathcal{N}^0 = \{\alpha\in\mathcal{N} \mid (\alpha,\hat{\delta}) \in 3\mathbb{Z}\}$$

is a sublattice of index 3 without roots. Note that $(\alpha_1,0,\dots,0) + \frac{1}{3}\hat{\delta}$ has norm 4 and the lattice $\mathcal{N}^0 + \mathbb{Z}((\alpha_1,0,\dots,0)+\frac{1}{3}\hat{\delta})$ is even unimodular without roots. Hence, it is isometric to the Leech lattice Λ (see Chapter 24 of [CS]).

Next, we construct some $\sqrt{2}E_8$ sublattices of $\mathcal{N}^0 < \Lambda$. Note that

$$\{(0^4, c, c) \mid c \in \mathcal{C}_4\} < \mathcal{C}_{12}.$$

Thus

$$\tilde{R} := \operatorname{span}\{(0,\ \gamma_c + z,\ \gamma_c + z) \mid z \in A_2^4,\ c \in \mathcal{C}_4\} < \mathcal{N}.$$

Since $E := \bigcup_{c\in\mathcal{C}_4}(\gamma_c + A_2^4) \simeq E_8$ it follows that $\tilde{R} \simeq \sqrt{2}E_8$.

Let $E^1 := 0 \oplus E \oplus 0$ and $E^2 := 0 \oplus 0 \oplus E$. Then $(E^1, E^2) = 0$ and $E^1 + E^2 < \frac{1}{3}\Lambda$.

For a codeword $x \in \mathcal{C}_{12}$, let $h(x) := \bar{\tau}^{x_1} \oplus \cdots \oplus \bar{\tau}^{x_{12}}$ where $\bar{\tau}$ is the previously defined isometry of A_2. Then $h(x)$ is an isometry of $\mathcal{N}$ and Λ [CS].

Consider now the codewords

$$d^1 = (0, -1, -1, -1,\ 0, 0, 0, 0,\ 0, 1, 1, 1) \quad \text{and} \quad d^2 = (1, 1, 0, -1,\ 0, 0, 0, 0,\ 1, 1, -1, 0)$$

of $\mathcal{C}_{12}$. Define $\hat{h}_1 := h(d^1)$ and $\hat{h}_2 := h(d^2)$. Note that $\hat{h}_1$ and $\hat{h}_2$ act on $E^1 + E^2$ as $\mathrm{id} \oplus h_1$ and $\mathrm{id} \oplus h_2$, where $h_1 = \mathrm{id} \oplus \bar{\tau} \oplus \bar{\tau} \oplus \bar{\tau}$, and $h_2 = \bar{\tau} \oplus \bar{\tau} \oplus \bar{\tau}^{-1} \oplus \mathrm{id}$ in $O(E_8)$ as previously defined. Then

$$\tilde{R^1} = \hat{h}_1(\tilde{R}) = \{(0, \alpha, \tilde{h}_1\alpha) \mid \alpha \in E^1\},$$
$$\tilde{R^2} = \hat{h}_2(\tilde{R}) = \{(0, \alpha, \tilde{h}_2\alpha) \mid \alpha \in E^1\}$$

are contained in $\mathcal{N}^0 < \Lambda$. It is also clear that $\tilde{L} = \tilde{R} + \tilde{R^1} + \tilde{R^2}$ is isometric to $L = R + R^1 + R^2$.

Remark A.4. It is known that a 3-transposition group generated by three involutions t_1, t_2, t_3 such that $t_3 \notin \langle t_1, t_2\rangle$ and any two of them generate S_3 is either isomorphic to S_4, 3^{1+2}:2 or 3^2:2 (see Lemma 2.5 of [CH1]). By Remark 3.9, the τ-involutions associated to $\{\hat{\sigma}^{-1}e_R,\ (\mathrm{id} \otimes \tilde{h}_1)\hat{\sigma}^{-1}e_R,\ (\mathrm{id} \otimes \tilde{h}_2)\hat{\sigma}^{-1}e_R\}$ generate a group of the shape 3^2:2 in $\mathrm{Aut}(V_\Lambda^+)$ and in $\mathrm{Aut}(V^\natural)$. Finally, we note that the centralizer of a 3C-element in Co_0 has the shape 3^{1+4}:$\mathrm{Sp}_4(3) \times 2$. It has a natural subgroup 3^{1+2}:2 which is generated by involutions with trace -8 on the Leech lattice (see (10.35.3) of [G2] or [ATLAS]). Recall that an involution of trace -8 on Λ has the fixed sublattice isometric to $\sqrt{2}E_8$ (cf. Theorem (10.15) of [G2] or Chapter 10 of [CS]). Thus, by formula (3.11), one can construct Ising vectors in $V_\Lambda^+ < V^\natural$ such that the corresponding Miyamoto involutions generate a group of the shape 3^{1+2}:2 in the Monster and the product of any two of them is in the conjugacy class 3A. In this case, the sub-VOA generated by the corresponding Ising vectors will not be isomorphic to $M_{\mathcal{C}_4}$ but the authors do not know the exact structure of such a VOA.

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