Generalized Moonshine for the Baby Monster

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Abstract

Nortons generalized Moonshine conjecture for the case related to the Baby Monster is proven. The main steps of the proof are the construction of the Baby Monster Lie algebra with the help of a vertex algebra, the use of the BRST-cohomology construction of bosonic string theory to identify the denominator identity of the constructed Lie algebra and the computation of the first 25 coefficients of the generalized Thompson series with the help of the equivariant denominator identity.

1 Introduction

1.1 Background

Monstrous Moonshine [Ogg75, Tho79, CN79] refers to some amazing connections between the Monster, the largest sporadic simple group [Gri82], and the theory of modular functions. In particular, Conway and Norton conjectured in [CN79] that there is a natural graded Moonshine module (cf. [FLM84]) $V^\natural = V^\natural_{(-1)} \oplus V^\natural_{(1)} \oplus V^\natural_{(2)} \oplus \cdots$ acted on by the Monster such that for every element g of the Monster the graded character

$$T_g = \mathrm{Tr}(g \mid V^\natural) = \sum_n \mathrm{Tr}(g \mid V^\natural_{(n)})\, q^n,$$

called the Thompson series, is the normalized generator of a genus zero function field arising from a naturally parametrized modular group Γ_g between some $\Gamma_0(N)$ and its normalizer in $\mathrm{PSL}_2(\mathbf{R})$. For g the identity, one has

$$T_1 = j - 744 = q^{-1} + 196884\, q + 21493760\, q^2 + 864299970\, q^3 + \cdots,$$

the classical elliptic modular function for the modular group $\mathrm{SL}_2(\mathbf{Z})$. Borcherds [Bor86] conjectured that the Moonshine module can be realized as a vertex operator algebra of central charge 24 with the Monster as automorphism group.

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A vertex operator algebra with the Monster as automorphism group was constructed by Frenkel, Lepowsky and Meurmann in [FLM88]. In [Bor92], Borcherds proved that all Thompson series associated to this vertex operator algebra are indeed Hauptmoduls for the genus zero function fields conjectured by Conway and Norton. The main steps in his proof are the construction of a rank 2 generalized Kac-Moody Lie algebra $\mathbf{g}^\natural$ acted on by the Monster, the use of the no-ghost from string theory to compute the root multiplicities of $\mathbf{g}^\natural$, and finally the examination of the equivariant denominator identity of $\mathbf{g}^\natural$ to identify the T_g by certain recursion formulas, called replicability formulas in [CN79].

However, not all monstrous moonshine phenomena are fully understood yet, cf. [Bor01, McK02]. The identification of the Monster as the sporadic simple group constructed in finite group theory with the automorphism group of the moonshine vertex operator algebra is quite indirect. Ogg's observation [Ogg75] that the primes p dividing the order of the monster are *exactly* those for which the function field determined by the normalizer of $\Gamma_0(p)$ in $\mathrm{PSL}_2(\mathbf{R})$ has genus zero, seems still unexplained. The Thompson series of the shorter Moonshine module constructed in [Höh95] which has up to a factor $\mathbf{Z}_2$ the Baby monster as automorphism group [Höh02] has not been determined until now. In [Nor87], Norton formulated a generalization of the original moonshine conjectures: For every element g of the Monster, he postulated the existence of a rational graded *twisted moonshine module* $V^\natural(g)$ acted on by (a central extension with $\mathbf{C}^*$) the centralizer $C_{\mathbb{M}}(g)$ of g such that for $h \in C_{\mathbb{M}}(g)$ the generalized Thompson series

$$T_{g,h}(\tau) = \mathrm{Tr}(h \mid V^\natural(g)) = \sum_n \mathrm{Tr}(g \mid V^\natural_{(n)})\, q^n, \quad q = e^{2\pi i \tau},$$

is either a constant or its invariance group $\Gamma_{g,h}$ in $\mathrm{PSL}_2(\mathbf{R})$ will again be a modular group of genus zero. Furthermore, $T_{g,h}$ depends only on the simultaneous conjugacy class of a commuting pair (g,h) of elements in the Monster and one has $T_{g^a h^c, g^b h^d}(\tau) = \gamma \cdot T_{g,h}(\frac{a\tau+b}{c\tau+d})$ with a root of unity γ for all pairs (g,h) and matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$.

Besides the genus zero property for $\Gamma_{g,h}$, all properties of the generalized Thompson series can easily be explained from properties of orbifolds in conformal field theory originally studied in mathematical physics, cf. [DVVV89]. The mathematical theory of orbifolds of vertex operator algebras has been developed in great parts by Dong, Mason and Li, see [DM96, DLM00]. In the cases where (g,h) can be mapped to $(g^a h^c, g^b h^d) = (1, f)$ by a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ their results imply the genus zero property of $\Gamma_{g,h}$, but the general case remains open.

1.2 This work

In this paper, we will prove the generalized Moonshine conjecture for the case of the Baby Monster $\mathbb{B}$, i.e., g an involution t of the Monster with centralizer $\mathrm{Cent}_{\mathbb{M}}(t) \cong 2.\mathbb{B}$, the nontrivial double cover of $\mathbb{B}$. The Baby Monster is the second largest sporadic simple group and has the order $2^{41}3^{13}5^6 7^2 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47$.

Main Theorem: *Let t be the involution of the Monster group with centralizer $2.\mathbb{B}$. Let*

$$V^\natural(t)=V^\natural_{(-1/2)}(t)\oplus V^\natural_{(1/2)}(t)\oplus V^\natural_{(1)}(t)\oplus V^\natural_{(3/2)}(t)\oplus\cdots$$

be the corresponding t-twisted module and denote for $h\in \mathrm{Cent}_{\mathbb{M}}(t)\cong 2.\mathbb{B}$ with

$$T_{t,h}(\tau)=\mathrm{Tr}(h\mid V^\natural(h))=\sum_n \mathrm{Tr}(h\mid V^\natural_{(n)})\,q^n,\quad q=e^{2\pi i\tau}$$

the generalized Thompson series. Then the functions $T_{t,h}(2\tau)$ are Hauptmoduls for genus zero modular groups $\Gamma_{t,h}$ of $\mathrm{PGL}_2(\mathbf{Q})$ containing some $\Gamma_0(N)$ as normal subgroups.

We give an outline of our proof.

Section 2: Let $V^\natural=V^{00}\oplus V^{01}$ respectively $V^\natural(t)=V^{10}\oplus V^{11}$ be the decomposition of $V^\natural$ resp. $V^\natural(t)$ into the $+1$ and -1 eigenspaces for $t\in\mathrm{Cent}_{\mathbb{M}}(t)$. One has:

The direct sum $V=V^\natural\oplus V^\natural(t)=\bigoplus_{(i,j)\in\mathbf{Z}_2\times\mathbf{Z}_2}V^{ij}$ has the structure of an abelian intertwining algebra for the abelian group $A=\mathbf{Z}_2\times\mathbf{Z}_2$, on which the group $2.\mathbb{B}$ acts by automorphisms.

Let $II_{1,1}(2)$ be the hyperbolic plane with all norms rescaled by the factor 2 and let $II_{1,1}'(2)\cong II_{1,1}(\frac{1}{2})$ be the dual lattice. The usual lattice type vertex algebra constructions again define on the space $V_{II_{1,1}'(2)}=\bigoplus_{\gamma\in II_{1,1}'(2)/II_{1,1}(2)}V_{II_{1,1}(2)+\gamma}=\bigoplus_{(i,j)\in\mathbf{Z}_2\times\mathbf{Z}_2}M^{ij}$ the structure of an abelian intertwining algebra for the group $\mathbf{Z}_2\times\mathbf{Z}_2$. Combining both intertwining algebras, we construct a vertex algebra structure of central charge 26 on the direct sum $W=\bigoplus_{(i,j)\in\mathbf{Z}_2\times\mathbf{Z}_2}V^{ij}\otimes M^{ij}$ together with an action of the group $2.\mathbb{B}$ by automorphisms.

Section 3: The Baby Monster Lie algebra $\mathbf{g}^\natural_{\mathbb{B}}$ is defined as follows: On the tensor product of the vertex algebra W with the bosonic ghost vertex superalgebra V_{ghost} of central charge -26, there is the action of the BRST-operator Q such that $Q^2=0$. The Lie algebra $\mathbf{g}^\natural_{\mathbb{B}}$ is defined as the corresponding BRST-cohomology group $H^1_{\mathrm{BRST}}(W)$ (cf. [FGZ86, Zuc89]). The Baby Monster Lie algebra $\mathbf{g}^\natural_{\mathbb{B}}$ is a $II_{1,1}'(2)$ graded Lie algebra of rank 2 on which $2.\mathbb{B}$ acts by automorphisms. Instead of the version of the no-ghost theorem from string theory as given in [Bor92] which identifies the physical states of the old covariant approach with the physical states in the light-cone model, we use the semi-infinite cohomology for the Virasoro algebra to compute the graded dimensions of the BRST-cohomology group in the BRST-approach, i.e. the root space multiplicities of $\mathbf{g}^\natural_{\mathbb{B}}$. Borcherds singular theta-correspondence applied to the vector valued modular function $f^a=\mathrm{Tr}(V^a)$, $a\in\mathbf{Z}_2\times\mathbf{Z}_2$, and the lattice $II_{1,1}(2)\oplus II_{1,1}$ leads to an automorphic product which can be identified with one side of the denominator identity of $\mathbf{g}^\natural_{\mathbb{B}}$. The Fourier expansion of the automorphic product allows us to identify all real and imaginary simple roots of $\mathbf{g}^\natural_{\mathbb{B}}$, cf. [Bor98]:

Theorem: *The denominator identity of* $\mathbf{g}^\natural_{\mathbb{B}}$ *is*

$$p^{-1/2}\prod_{\substack{m\in\frac{1}{2}\mathbf{Z},\, m>0\\ n\in\frac{1}{2}\mathbf{Z}}}(1-p^mq^n)^{c^{[m,n]}(2mn)}=f(p)-f(q)$$

where $f(q)=f^{10}(q)+f^{11}(q)=q^{-1/2}+95256\,q^{1/2}+\cdots$, *the coefficient of* q^n *in* $f^{ij}(q)$ *is denoted by* $c^{ij}(n)$ *and* $[m,n]$ *is the residue class of* $2(m,n)\in\mathbf{Z}\oplus\mathbf{Z}$ *modulo* 2.

It turns out, that our Baby Monster Lie algebra is isomorphic to the one defined by Borcherds via generator and relations from the t-twisted denominator identity of the Monster Lie algebra $\mathbf{g}^\natural$. The important feature of our construction is that we have the non-trivial action of $2.\mathbb{B}$ by automorphisms.

Section 4: From the equivariant denominator identity

$$\mathrm{Tr}(h\mid\Lambda^*(E))=\mathrm{Tr}(h\mid H^*(E))$$

it follows that all generalized Thompson series $T_{t,h}(2\tau)=q^{-1}+\sum_{i=1}^{\infty}a_h(i)\cdot q^i$ are replicable functions in the sense of [Nor84]. Since such functions are determined by the first 23 coefficients $a_h(i)$, $1\leq i\leq 23$, it is enough to compute those and secondly to show that the resulting functions are indeed Hauptmoduls for genus zero modular groups.

The first part can be worked out with the help of the Thompson series for the Moonshine module. For the first coefficients up to $a_h(7)$, this has essentially been done in [Höh95], Lemma 4.2.10. Together with extra recursion formulas obtained from the denominator identity, the remaining one can also be computed. On the other side, Norton [Nor01] has found a conjecturally complete list of replicable functions with rational coefficients and all nontrivial ones are indeed Hauptmoduls. It turns out that all of our functions $T_{t,h}(2\tau)$ are in his list.

As corollaries, we determine the Thompson series of the shorter Moonshine module $VB^\natural$ in term of the series $T_{t,h}$ and show that the even vertex operator subalgebra $VB^\natural(0)$ belongs to the class $\mathcal{S}^6$ of vertex operator algebras as defined by Matsuo [Mat01]

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2 The vertex algebra for the Baby Monster

In this section, we define a vertex algebra W of central charge 26 by glueing together the fixpoint vertex operator algebra $(V^\natural)^t$ of central charge 24 with the lattice vertex algebra $V_{II_{1,1}(2)}$ of central charge 2.

2.1 The vertex operator algebra $(V^\natural)^t$

Let $V^{00} = (V^\natural)^t$ respectively V^{01} be the $+1$- respectively -1-eigenspace for the involution t of the Monster group $\mathrm{Aut}(V^\natural)$ which has the two-fold cover $2.\mathbb{B}$ of the Baby Monster as centralizer.

The Moonshine module has only itself as irreducible module [Don94]. For such a vertex operator algebra V it is known [DLM00, DLM95] that there exist for an element $g \in \mathrm{Aut}(V)$ of finite order a unique twisted module $V(g)$ on which a central extension $\widetilde{\mathrm{Aut}}(V(g))$ of the centralizer of g in $\mathrm{Aut}(V)$ by $\mathbf{C}^*$ acts by module automorphisms, i.e. one has

$$h\, Y_{V(h)}(v,z)\, h^{-1} = Y_{V(h)}(\bar{h}v, z)$$

for elements $h \in \widetilde{\mathrm{Aut}}(V)$ with image $\bar{h} \in \mathrm{Cent}_{\mathrm{Aut}(V)}(g)$. Applying this to the Moonshine module $V^\natural$ and the involution t, we get an action of a $\mathbf{C}^*$-extension of $2.\mathbb{B}$ on $V^\natural(t)$ by module automorphisms. Since $H^2(\mathbb{B}, \mathbf{C}^*) \cong \mathbf{Z}_2$, the $\mathbf{C}^*$-extension of $2.\mathbb{B}$ is split and by choosing such a split we get an action of $2.\mathbb{B}$ on $V^\natural(t)$. Let $V^{10} = (V^\natural(t))^t$ respectively V^{11} be the $+1$- respectively -1-eigenspace of the central involution $t \in 2.\mathbb{B}$.

From the definition of twisted modules, it follows that V^{00}, V^{01}, V^{10} and V^{11} are V^{00}-modules. We have an action of $2.\mathbb{B}$ by module automorphisms on each of them.

The structure theory of framed vertex operator algebras allows us to understand the representation theory of $(V^\natural)^t$. The rational Virasoro VOA $L_{1/2}$ of central charge $1/2$ has three irreducible modules $L(0)$, $L(1)$ and $L(2)$ of conformal weight 0, $1/2$ and $1/16$, respectively. The irreducible modules of $L_{1/2}^{\otimes 24}$ are the tensor products $L(i_1) \otimes \cdots \otimes L(i_{24})$, $i_1, \ldots, i_{24} \in \{0, 1, 2\}$, for which we write shortly $L(i_1, \ldots, i_{24})$. W recall the definition of a framed vertex operator algebras from [DGH98]. A subset $F = \{\omega_1, \ldots, \omega_r\}$ of a simple vertex operator algebra V is called a *Virasoro frame* if the ω_i for $i = 1, \ldots, r$ generate mutually commuting simple Virasoro vertex operator algebras $L_{1/2}$ of central charge $1/2$ and $\omega_1 + \cdots + \omega_r$ is the Virasoro element of V. Such a vertex operator algebra V is called a *framed vertex operator algebra*. Under $L_{1/2}^{\otimes r}$, the vertex operator subalgebra spanned by the Virasoro frame, V decomposes as

$$V = \bigoplus_{h_1, \ldots, h_r \in \{0, \frac{1}{2}, \frac{1}{16}\}} n_{h_1, \ldots, h_r}\, L(h_1, \ldots, h_r),$$

into isotypical components of $L_{1/2}^{\otimes r}$-modules.

Theorem 2.1 *The VOA $(V^\natural)^t$ has four isomorphy types of irreducible modules V^{ij}. The fusion algebra of the above $(V^\natural)^t$-modules is isomorphic to the group ring of $\mathbf{Z}_2 \times \mathbf{Z}_2$. Furthermore, the direct sum $V^\natural \oplus V^\natural(t) = V^{00} \oplus V^{01} \oplus V^{10} \oplus V^{11}$ of $(V^\natural)^t$-modules can be given the structure of an abelian intertwining algebra for the group $A = \mathbf{Z}_2 \times \mathbf{Z}_2$ and associated quadratic form $\Omega(g,g)$ with the values $\Omega((0,0),(0,0)) = 1$, $\Omega((0,1),(0,1)) = 1$, $\Omega((1,0),(1,0)) = 1$ and $\Omega((1,1),(1,1)) = -1$.*

Proof. From the decomposition of the Moonshine module as Virasoro module given in [DGH98], Sect. 5, we get the Virasoro module decomposition of the V^{00}-modules V^{00} and V^{01} for a distinguished Virasoro frame of V^{00}. The structure of the twisted module $V^\natural(t)$ from the viewpoint of framed vertex operator algebras has been investigated by C. Lam [Lam00]. See also Yamauchi [Yam02]. The rest of the proof is analog to [Hua96], where a intertwining algebra on the direct sum of the Moonshine module and its $2B$-twisted module was given, just by using the module structure for the Virasoro frame. $\square$

Let $\mathcal{Y}$ be an intertwining operator between simple V^{00}-modules. For $h \in 2.\mathbb{B}$ the map $h^{-1}\mathcal{Y}(hv,z)h$ is again an intertwining operator.

Lemma 2.2 *The action of $2.\mathbb{B}$ on the space of intertwining operators between simple $(V^\natural)^t$-modules is trivial.*

Proof. The above assignment leads to a representation of $2.\mathbb{B}$ on the intertwining spaces. Since all the intertwining spaces in question are one-dimensional or zero, the representation of $2.\mathbb{B}$ must be trivial. $\square$

The action is therefor compatible with the intertwining algebra structure on $V^{00} \oplus V^{01} \oplus V^{10} \oplus V^{11}$:

Theorem 2.3 *The action of $2.\mathbb{B}$ on $V^{00} \oplus V^{01} \oplus V^{10} \oplus V^{11}$ is an automorphism of the abelian intertwining algebra structure in the sense that for $g \in 2.\mathbb{B}$ one has*

$$g\mathcal{Y}(v,z)g^{-1} = \mathcal{Y}(gv,z)$$

for all intertwining operators $\mathcal{Y} : V^{ij} \times V^{i'j'} \longrightarrow V^{i+i',j+j'}$ and all pairs (i,j), (i',j') in $\{(0,0),(0,1),(1,0),(1,1)\}$. $\square$

We relate the above intertwining algebra structure with our construction of the shorter Moonshine module in [Höh95]. The shorter Moonshine module $VB^\natural$ is a vertex operator superalgebra of central charge $23\frac{1}{2}$ having the conformal character

$$\chi_{VB^\natural} = q^{-47/48}(1 + 4371\,q^{3/2} + 96256\,q^2 + 1143745\,q^{5/2} + \cdots).$$

We recall its definition from [Höh95], Sec. 3.1 and 4.2: Let $L_{1/2}$ be the Virasoro vertex operator subalgebra of $V^\natural$ generated by a transposition axis e. Let $VB^\natural(0)$ be the commutant of $L_{1/2}$ in $V^\natural$. One has a decomposition

$$V^\natural \;=\; VB^\natural(0)\otimes L(0) \;\oplus\; VB^\natural(1)\otimes L(1) \;\oplus\; VB^\natural(2)\otimes L(2)$$

with $VB^\natural(0)$-modules $VB^\natural(0)$, $VB^\natural(1)$ and $VB^\natural(2)$. The shorter Moonshine module is defined as $VB^\natural = VB^\natural(0)\oplus VB^\natural(1)$. A vertex operator superalgebra structure on $VB^\natural$ such that $VB^\natural(0)$ resp. $VB^\natural(1)$ is the even resp. odd part can be constructed from the vertex operator algebra structure of $V^\natural$ (see [Höh95], Th. 3.1.3). The centralizer of the class $2A$ involution associated to the idempotent e is $2.\mathbb{B}$. There is an induced an action of $2.\mathbb{B}$ on $VB^\natural$ respecting the vertex operator superalgebra structure such that the central involution acts trivial (see [Höh95], Th. 4.2.6). An extra automorphism of $VB^\natural$ commuting with the $\mathbb{B}$-action is the involution which acts as the identity on $VB^\natural(0)$ and by multiplication with -1 on $VB^\natural(1)$. The resulting group $\mathbf{Z}_2\times\mathbb{B}$ is the complete automorphism group of $VB^\natural$ [Höh02].

As $VB^\natural(0)\otimes L_{1/2}(0)$-modules, one has the decompositions

$$\begin{aligned}(V^\natural)^t = V^{00} &= VB^\natural(0)\otimes L(0)\oplus VB^\natural(1)\otimes L(1)\\ V^{01}=V^{10} &= VB^\natural(2)\otimes L(2)\\ V^{11} &= VB^\natural(0)\otimes L(1)\oplus VB^\natural(1)\otimes L(0)\end{aligned}\tag{1}$$

as one can see from the Virasoro module structure for the Virasoro frame in V^{00}. Together with the trivial $2.\mathbb{B}$-action on the $L_{1/2}(0)$-modules $L(i)$, $i=0$, 1 and 2, the resulting action V^{ij} using the decompositions is isomorphic to the $2.\mathbb{B}$-actions on the V^{ij}.

There is the grading on a Vertex operator algebra or vertex operator algebra module module $V=\bigoplus_{n\in\frac{1}{l}\mathbf{Z}} M_n$ given by the eigenvalues of the Virasoro generator L_0. The *conformal character* of M defined by

$$\chi_M = q^{-c/24}\sum_{n\in\mathbf{Q}}\dim V_n\, q^n$$

where c is the central charge is an element in the ring of formal Laurent series in $q^{1/k}$ and $q^{c/24}$.

For the proof of some identities, it is useful to interpret an element f in $\mathbf{C}[[q^{1/k}]][q^{-1/k}]$, where L is a lattice and $k\in\mathbf{N}$, as a function on the complex upper half plane $\mathcal{H}=\{z\in\mathbf{C}\mid \mathrm{Im}(z)>0\}$. This is done by the substitutions $q\mapsto e^{2\pi i\tau}$ for $\tau\in\mathcal{H}$ (in the case of convergence). We indicate this by writing $f(\tau)$.

Let $\eta(\tau)=q^{1/24}\prod_{n=1}^\infty(1-q^n)$ be the Dedekind eta function.

The graded characters $\chi_i=\chi_{L(i)}$ of the three irreducible representations $L(i)$, $i=0$, 1 and 2, of the Virasoro vertex algebra of central charge 1/2 are modular functions weight 0. They can be expressed through η-products as follows:

$$\chi_0(q) \;=\; \chi_{L(0)}(q)=q^{-1/48}\cdot\frac{1}{2}\left(\prod_{n=0}^\infty(1+q^{n+1/2})+\prod_{n=0}^\infty(1-q^{n+1/2})\right)$$

$$
\begin{aligned}
\chi_1(q) &= \chi_{L(1)}(q) = q^{-1/48}\cdot \frac{1}{2}\left(\prod_{n=0}^{\infty}(1+q^{n+1/2})-\prod_{n=0}^{\infty}(1-q^{n+1/2})\right) \qquad (2)\\
\chi_2(q) &= \chi_{L(2)}(q) = q^{-1/48+1/16}\cdot \prod_{n=1}^{\infty}(1+q^{n}).
\end{aligned}
$$

The characters of the $(V^\natural)^t$-modules V^{ij} are easily obtained from the Virasoro decomposition.

Theorem 2.4 *Let*

$$
f(q)=\frac{\eta(q^{1/2})^{24}}{\eta(q)^{24}}+24+2^{12}\frac{\eta(q)^{24}}{\eta(q^{1/2})^{24}}=q^{-1/2}+96256\,q^{1/2}+\cdots.
$$

$f(q^2)$ is the normalized Hauptmodul for the modular group $\Gamma_0(2)+$, the normalizer of $\Gamma_0(2)$ in $\mathrm{PSL}_2(\mathbf{R})$. The conformal characters of the four irreducible of $(V^\natural)^t$-modules described in Theorem 2.1 are the following modular functions:

$$
\begin{aligned}
\chi_{V^{00}}(q) &= f^{00}(q)=f(q^2)+\tfrac{1}{2}\big(f(q)+f(-q)\big)\\
&= q^{-1}+100628\,q+10795008\,q^2+432769986\,q^3+10128277504\,q^4+\cdots\\
\chi_{V^{01}}(q) &= f^{01}(q)=\tfrac{1}{2}\big(f(q)+f(-q)\big)=\\
\chi_{V^{10}}(q) &= f^{10}(q)\\
&= 96256\,q+10698752\,q^2+431529984\,q^3+10117578752\,q^4+\cdots\\
\chi_{V^{11}}(q) &= f^{11}(q)=\tfrac{1}{2}\big(f(q)-f(-q)\big)\\
&= q^{-1/2}+4372\,q^{1/2}+1240002\,q^{3/2}+74428120\,q^{5/2}+2206741887\,q^{7/2}+\cdots
\end{aligned}
$$

Proof. We use the decompositions (1), formulas (2) and [Höh95], Th. 4.2.8, where the characters of the $VB^\natural(i)$, $i=0$, 1 and 2 have been computed with the help of the Virasoro module decomposition. The argument is that the space of modular functions for the appropriate modular group with poles of fixed order only at the cusps is finite-dimensional. Comparing the Fourier expansions on both sides gives the result. □

2.2 The lattice vertex algebra V_L

Denote the dual lattice of an integral lattice N by N'. We write $N(k)$ for the lattice obtained from N by rescaling all norms by a factor k.

The fusion algebra of a vertex operator algebra V_N associated to an even integral lattice N can be identified with the group ring $\mathbf{C}[N'/N]$. The conformal weights modulo $\mathbf{Z}$ of the irreducible V_N-modules $V_{N+\gamma}$, where γ runs through a complete system of coset representatives for N'/N, can be identified with the quadratic discriminant form on N'/N. On the direct sum of the irreducible modules of a lattice type vertex operator algebra there is the structure of an abelian intertwining algebra [DL93], Th. 12.24 such that the cohomology class of the associated 3-cocycle is determined by the discriminant form on N'/N.

Let $L = II_{1,1}(2)$ be the Hyperbolic plane with all norms rescaled by the factor 2 and let $L' = II_{1,1}(2)' \cong II_{1,1}(\frac{1}{2})$ be the dual lattice. If we write $L' = \frac{1}{2}\mathbf{Z} \oplus \frac{1}{2}\mathbf{Z}$ and use the rational base vectors $(1,0)$ and $(0,1)$, the bilinear form is given by the matrix $\left(\begin{smallmatrix} 0 & -2 \\ -2 & 0 \end{smallmatrix}\right)$. The norm of $(m,n) \in \frac{1}{2}\mathbf{Z} \oplus \frac{1}{2}\mathbf{Z}$ is $-4mn$.

Theorem 2.5 *The direct sum of modules of V_L has the structure of an abelian intertwining algebra for the discriminant group L'/L. The associated quadratic form Ω can be identified with the discriminant form on L'/L induced from the norm on L'.* $\square$

Proof. For the case of definite lattices, see [DL93], Th. 12.24. For the case of indefinite lattices which we will need here, see [Sch00], Th. 2.2. $\square$

The form Ω takes the values 1, 1, 1 and -1 on the 4 cosets L'/L, with -1 on the coset belonging to those $(m,n) \in L'$ where m and n are both non-integral.

2.3 The vertex algebra W

Definition 2.6 Fix an isometry $\iota : A \longrightarrow L'/L$. Let

$$W = \bigoplus_{(i,j)\in A} V^{ij} \otimes V_{L+\iota((i,j))}.$$

Proposition 2.7 *The space W has a unique simple vertex algebra structure extending the $(V^\natural)^t \otimes V_L$-module structure.*

Proof. The tensor product of the two intertwining algebras $(V^\natural)^t \oplus V^\natural(t)$ and $V_{L'}$ is an intertwining algebra for the group $A \times L'/L$ and quadratic form $q : A \times L'/L \longrightarrow \{+1,-1\}$, $(a,[\gamma]) \mapsto \Omega(a,a)\cdot\Omega([\gamma],[\gamma])$; see [DL93], Prop. 12.39 (with an obvious generalization to vertex algebras). The subgroup $U = \{(a,\iota(a)) \mid a \in A\} \subset A \times L'/L$ is isotropic for q. It was shown in Theorem 4.3 of [Höh01] that the induced intertwining algebra structure on W can be adjusted in an up to isomorphism unique way to a simple vertex algebra structure. $\square$

By taking the trivial $2.\mathbb{B}$-action on the irreducible V_L-modules $V_{L+\gamma}$, $[\gamma] \in L'/L$, the following results follows from Theorem 2.3:

Theorem 2.8 *The two-fold cover $2.\mathbb{B}$ acts by automorphisms on the vertex algebra W, respecting the grading by the lattice L' and the $\mathbf{Z}$-grading by the conformal weight obtained from the Virasoro module structure.* $\square$

3 The Baby Monster Lie algebra $\mathbf{g}_{\mathbb{B}}^{\natural}$

In this section, we use the BRST-cohomology construction from string theory to construct a generalized Kac-Moody algebra $\mathbf{g}_{\mathbb{B}}^{\natural}$ from W. There is an action of the BRST-operator on the tensor product of a vertex algebra W of central charge 26 with the bosonic ghost vertex superalgebra V_{ghost} of central charge -26, which defines the BRST-cohomology groups $H^*_{\mathrm{BRST}}(W)$. The degree 1 cohomology group $H^1_{\mathrm{BRST}}(W)$ has additionally the structure of a Lie algebra, see [FGZ86, LZ93]. Theorem 2.4 allows us to describe its root multiplicities. We determine the simple roots with the help of the singular theta-correspondence.

The following construction was outlined in [Zuc89], section 4.

The bosonic ghost vertex superalgebra V_{ghost} is defined as the vertex superalgebra for the integer lattice $\mathbf{Z}\sigma$, $(\sigma,\sigma)=1$, together with the Virasoro element

$$\omega^{\mathrm{Gh}} = \tfrac{1}{2}\sigma(-1)_{-1}\sigma(-1) + \tfrac{3}{2}\sigma(-2)$$

generating representations of the Virasoro algebra of central charge -26. One introduces the states

$$b = e^{-\sigma} \qquad \text{and} \qquad c = e^{\sigma}.$$

The modes of the corresponding vertex operators generate an infinite-dimensional Heisenberg superalgebra. The states b and c have L_0^{Gh} eigenvalues 2 and -1.

Let W be the vertex algebra of the last section. We study certain fields in the vertex algebra $W \otimes V_{\mathrm{ghost}}$. First, there is the Virasoro element

$$\omega = \omega^W + \omega^{\mathrm{Gh}}$$

leading to a representation of the Virasoro algebra of central charge 0. Let

$$j^{\mathrm{BRST}} = c_{-1}\omega^W + \tfrac{1}{2}c_{-1}\omega^{\mathrm{Gh}}$$

be the BRST-current. The zero mode j_0^{BRST} is called the BRST-operator and denoted by Q. We also need

$$j^N = \sigma(-1).$$

The zero mode j_0^N is called the ghost number operator and denoted by U. One has

$$Q^2 = 0, \qquad [U,Q] = Q, \qquad [Q,L_0] = 0, \qquad \text{and} \qquad [U,L_0] = 0 \tag{3}$$

and furthermore

$$[Q,b_1] = L_0. \tag{4}$$

The operators U and L_0 are commuting, act diagonally and respect the grading of $W = \bigoplus_{\alpha\in L'} W(\alpha)$ given by the lattice L'. We denote with $W(\alpha)_n^u$

the subspace of $W(\alpha)$ on which U and L_0 act by the eigenvalues u and n. From the equations (3) we see that we have a complex

$$\cdots \xrightarrow{Q} W(\alpha)_n^{u-1} \xrightarrow{Q} W(\alpha)_n^u \xrightarrow{Q} W(\alpha)_n^{u+1} \xrightarrow{Q} \cdots$$

leading to cohomology groups $H^*_{\mathrm{BRST}}(W) = \bigoplus_{\alpha,\,n} H^*(\alpha)_n$.

On $H^1_{\mathrm{BRST}}(W)$ there is the structure of a Lie algebra (cf. [LZ93], Th. 2.2).

Definition 3.1 The Baby Monster Lie algebra $\mathbf{g}_{\mathbb{B}}^{\natural}$ is defined as the Lie algebra $H^1_{\mathrm{BRST}}(W)$.

From the construction it is clear that $\mathbf{g}_{\mathbb{B}}^{\natural}$ is graded by the lattice L' and that the action of $2.\mathbb{B}$ on W leads to an action of $2.\mathbb{B}$ on $\mathbf{g}_{\mathbb{B}}^{\natural}$ respecting this grading.

For the identification of the components $\mathbf{g}_{\mathbb{B}}^{\natural}(\alpha) = \bigoplus_n H^1(\alpha)_n$, we define the relative BRST-complex: Let $B(\alpha)_n^u = W(\alpha)_n^u \cap \ker b_1$ and $C(\alpha)^u = B(\alpha)_0^u$. From equation (4) we see that we have a subcomplex

$$\cdots \xrightarrow{Q} C(\alpha)^{u-1} \xrightarrow{Q} C(\alpha)^u \xrightarrow{Q} C(\alpha)^{u+1} \xrightarrow{Q} \cdots$$

leading to the relative cohomology groups $H^*_{\mathrm{rel}}(C) = \bigoplus_{\alpha,\,n} H^*_{\mathrm{rel}}(\alpha)$.

As shown in [LZ89], the BRST- and the relative BRST-complex are connected by a short exact sequence of cochain complexes leading to the isomorphism $H^1_{\mathrm{BRST}}(W) = H^0_{\mathrm{rel}}(C)$. The relative BRST-cohomology groups can now be identified with the relative semi-infinite cohomology groups [FGZ86, LZ91] for the Virasoro algebra as introduced by Feigin [Fg84]:

$$H^u_{\mathrm{rel}}(\alpha) = H^u_{\infty/2}(\mathrm{Vir}, \mathrm{Vir}_0; V^{[\alpha]} \otimes H(2,\alpha)),$$

where $H(2,\alpha)$ is the highest weight module of highest weight α for the rank 2 infinite Heisenberg vertex algebra of Lorentzian signature. The vanishing theorem ([Fg84]; cf. [LZ91], Cor. 2.6) gives for $\alpha \neq 0$:

$$H^u_{\infty/2}(\mathrm{Vir}, \mathrm{Vir}_0; V^{[\alpha]} \otimes H(2,\alpha)) = 0 \qquad \text{for } u \neq 0.$$

It follows that $H^0_{\mathrm{rel}}(\alpha)$ can be determined from the $B(\alpha)_n^u$ by the Euler-Poincare principle and counting the states of conformal weight $n = 0$. From the proofs of Th. 3.5 and Cor. 2.27 of [LZ91] we get

$$\dim H^0_{\mathrm{rel}}(\alpha) = \text{Coefficient of } q^0 \text{ in } q^{(\alpha,\alpha)/2}\chi_{V^{[\alpha]}}(q).$$

For $\alpha = 0$ one finds $H^0_{\mathrm{rel}}(0) \cong \mathbf{R}^2$. We collect this information in the next proposition.

Proposition 3.2 *The Lie algebra $\mathbf{g}_{\mathbb{B}}^{\natural}$ is graded by the lattice L'. Its components $\mathbf{g}_{\mathbb{B}}^{\natural}(\alpha)$ for $\alpha \in L'$ are as $2.\mathbb{B}$-modules isomorphic to $V^{[\alpha]}_{1-\alpha^2/2} = V^{[\alpha]}_{(-\alpha^2/2)}$ for $\alpha \neq 0$ and to $V_1^{00} \oplus \mathbf{R}^2 \cong \mathbf{R}^2$ for $\alpha = 0$.*

Remark 3.3 The vertex algebra W has an invariant bilinear form which can be used to show that the definition of $\mathbf{g}_{\mathbb{B}}^{\natural}$ as BRST-cohomology group is equivalent to the so called old covariant construction used in [Bor92] (cf. [LZ93], section 2.4, cf. [Zuc89], section 4). In the old covariant approach, the Baby Monster Lie algebra $\mathbf{g}_{\mathbb{B}}^{\natural}$ is the Lie algebra $(P_1/L_{-1}P_0)/\mathrm{Rad}(\,,\,)$ where P_i denotes the space of Virasoro highest weight vectors of weight i in W, the Lie bracket is induced from the vertex algebra structure on W and $(\,,\,)$ comes from a natural bilinear form on W such that that the adjoint of the Virasoro generator L_n is L_{-n}. In particular, we see that $\mathbf{g}_{\mathbb{B}}^{\natural}$ posses an invariant non-degenerated bilinear form. Alternatively, one can directly construct an invariant bilinear form on the relative BRST-cohomology group $H^0_{\mathrm{rel}}(C)$ as in [Sch00].

We recall some results about generalized Kac-Moody algebras.

Let $A=(a_{ij})_{i,\,j\in I}$ be a real symmetric square matrix for some finite or countable index set I with the properties

$$\begin{aligned} a_{ij} \leq 0 \quad &\text{if} \quad i\neq j,\\ 2a_{ij}/a_{ii}\in \mathbf{Z} \quad &\text{if} \quad a_{ii}>0. \end{aligned} \tag{5}$$

Let $\hat{\mathbf{g}}$ be the real or complex Lie algebra with generators e_i, f_i, h_{ij} for $i,\,j\in I$ and relations

$$\begin{aligned} [e_i,f_j] &= h_{ij},\\ [h_{ij},e_k] &= \delta_i^j a_{ik}\,e_k,\\ [h_{ij},f_k] &= -\delta_i^j a_{ik}\,f_k,\\ (\mathrm{ad}\,e_i)^{1-2a_{ij}/a_{ii}}e_j &= (\mathrm{ad}\,f_i)^{1-2a_{ij}/a_{ii}}f_j \;=\; 0,\\ [e_i,e_j] &= [f_i,f_j] \;=\; 0 \;\text{ if } a_{ij}=0. \end{aligned}$$

The elements h_{ij} are zero unless the ith and jth column of A are equal. The nonzero h_{ij} are linearly independent and span an abelian subalgebra $\hat{H}$ of $\hat{\mathbf{g}}$. The elements h_{ii} are usually denoted h_i. Every nonzero ideal of $\hat{\mathbf{g}}$ has nonzero intersection with $\hat{H}$. The center of $\hat{\mathbf{g}}$ is in $\hat{H}$ and contains all the elements h_{ij} with $i\neq j$.

A Lie algebra $\mathbf{g}$ is called a generalized Kac-Moody algebra if $\mathbf{g}$ is the semi-direct product $(\hat{\mathbf{g}}/C)\cdot D$ where C is a subspace of the center of $\hat{\mathbf{g}}$ and D is an abelian subalgebra of $\mathbf{g}$ such that the elements e_i and f_i are all eigenvectors for D.

Let $\mathbf{g}$ now be a generalized Kac-Moody algebra. We keep the notion e_i, f_i, h_i for the images in the quotient (cf. [Bor95]).

The root lattice Q of $\mathbf{g}$ is the free abelian group generated by elements α_i, $i\in I$, and the bilinear form defined by $(\alpha_i,\alpha_j)=a_{ij}$. The elements α_i are called simple roots. Defining the degree of e_i as α_i, of f_i as $-\alpha_i$ and of h_i as zero, $\hat{\mathbf{g}}$ becomes Q-graded. We have the usual definitions of roots and root spaces. A root is called real if it has positive norm and imaginary else. The space $H = (\hat{H}/C) \oplus D$ is a commutative subalgebra of $\mathbf{g}$ called the Cartan subalgebra. There is a unique invariant symmetric bilinear form on $\mathbf{g}$ satisfying $(h_i, h_j) = a_{ij}$. We have a natural homomorphism of abelian groups from Q to H sending α_i to h_i. By abuse of terminology the images of roots under this map are also called roots. It is important to note that this map is in general not injective. It is possible that $n > 1$ imaginary simple roots have the same image h in H. In this case we call h a root with multiplicity n. If the matrix A has infinitely many identical columns this multiplicity may even be infinite.

A vector $\rho \in H$ is called a Weyl vector if one has $(\rho, \alpha) = -(\alpha, \alpha)/2$ for all simple roots α.

Borcherds gave the following theorem for identifying generalized Kac-Moody algebras.

Theorem 3.4 (Borcherds, [Bor95]) *Any Lie algebra* $\mathbf{g}$ *satisfying the following five conditions is a generalized Kac-Moody algebra:*

i) $\mathbf{g}$ *has a nonsingular invariant symmetric bilinear form* $(\ ,\ )$*.*

ii) $\mathbf{g}$ *has a self centralizing subalgebra* H *such that* $\mathbf{g}$ *is the sum of eigenspaces of* H *and all eigenspaces are finite dimensional.*

iii) The bilinear form restricted to $Q \otimes \mathbf{R} \subset H$ *is Lorentzian.*

iv) The norms of roots of $\mathbf{g}$ *are bounded above.*

v) If two roots are positive multiples if the same norm zero vector then their root spaces commute.

The character formula of Kac-Moody algebras can be generalized:

Theorem 3.5 (Borcherds-Kac-Weyl [Bor88]) *Let* V_λ *be the irreducible heighest weight module of integrable highest weight* λ *for a generalized Kac-Moody algebra* $\mathbf{g}$*. Then*

$$\chi(V_\lambda) = \frac{\sum_{w \in W} \operatorname{sgn}(w) w\left(\sum_S (-1)^{|S|} e^{\rho + \lambda + \sum_{\alpha \in S} \alpha}\right)}{e^\rho \prod_{\alpha > 0} (1 - e^{-\alpha})^{\operatorname{mult} \alpha}}$$

where W *is the Weyl group generated by the reflections at the real roots,* ρ *is a Weyl vector,* S *runs through all sets of mutually orthogonal imaginary simple roots and the product in the denominator is over all positive roots.*

In the special case $\lambda = 0$, i.e., $V_0 \cong \mathbf{C}$, we obtain the denominator identity of $\mathbf{g}$.

Now we study again the Baby Monster Lie algebra $\mathbf{g}^\natural_{\mathbb{B}}$.

Theorem 3.6 *Let* $f^{ij}(\tau) = \sum_{n\in\frac{1}{2}\mathbf{Z}} c^{ij}(n)\, q^n$ *be the Fourier expansion of the conformal characters of the modules* V^{ij}*,* $(i,j) \in A$ *given in Theorem 2.4. For a nonzero vector* $\alpha \in L'$ *the dimension of the root space component* $\mathbf{g}^\natural_{\mathbb{B}}(\alpha)$ *is*

$$\dim \mathbf{g}^\natural_{\mathbb{B}}(\alpha) = c^{[\alpha]}(-\alpha^2/2),$$

where $[\alpha]$ *denotes the rest class of* α *in* $L'/L = A$*. The dimension of the Cartan subalgebra of* $\mathbf{g}^\natural_{\mathbb{B}}$ *is* 2*.*

Proof. Theorem 2.4, Proposition 3.2. $\square$

Theorem 3.7 *The Baby Monster Lie algebra is a generalized Kac-Moody algebra.*

Proof. We will check that the five conditions of Theorem 3.4 are satisfied. Condition i) follows from Remark 3.3. If we take for H the subspace $\mathbf{g}(0)$ of degree $0 \in L'$ condition ii) (since a computation shows $[h,x] = (\alpha,h)x$ for $h \in H$ and $x \in \mathbf{g}^\natural_{\mathbb{B}}(\alpha)$) and iii) (by construction) are satisfied. We see from Prop 3.6 that the norm of all roots are bound above by 1, i.e. condition iv) is true. Finally, condition v) is checked again by an explizit computation. $\square$

We have summarized the roots space decomposition of the Baby Monster Lie algebra for small degrees $\alpha = (m,n) \in L'$ in Table 1. It follows from the Fourier expansion of the f^{ij} that the real roots of $\mathbf{g}^\natural_{\mathbb{B}}$ are the two vectors $\pm(1/2,-1/2)$ of L' of norm 1. The real roots of $\mathbf{g}^\natural_{\mathbb{B}}$ generate the Weyl group $W \cong \mathbf{Z}_2$ of $\mathbf{g}^\natural_{\mathbb{B}}$ which is also equal to the reflection group of L'. Both real roots are simple.

Lemma 3.8 *The functions* $(f^{ij})_{(i,j)\in A}$ *form a vector valued modular function of weight* 0 *with respect to the Weyl representation of* $SL_2(\mathbf{Z})$ *for the quadratic space* (A,q)*.*

Proof. From the explizit expression of the f^{ij}, $(i,j) \in A$, given in (3) in terms of η-products, it is easy to write down the S and T-matrix explicitly. $\square$

Proposition 3.9

$$p^{-1/2} \prod_{\substack{m\in\frac{1}{2}\mathbf{Z},\, m>0 \\ n\in\frac{1}{2}\mathbf{Z}}} (1-p^m q^n)^{c^{[m,n]}(2mn)} = f(p) - f(q)$$

where $f(q) = f^{10}(q) + f^{11}(q) = q^{-1/2} + 95256\, q^{1/2} + \cdots$*.*

Proof. This is Example 13.9 in Borcherds [Bor98], a special case of the theory of theta lifts for vector valued modular functions developed there. The formula is the t-twisted denominator identity of the Monster Lie algebra. $\square$

Table 1: The root space structure of the Baby Monster Lie algebra

$m\backslash n$	$\cdots$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\cdots$
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
3	$\cdots$	0	0	0	$V^{10}_{(3)}$	$V^{00}_{(6)}$	$V^{10}_{(9)}$	$V^{00}_{(12)}$	$V^{10}_{(15)}$	$V^{00}_{(18)}$	$\cdots$
$\frac{5}{2}$	$\cdots$	0	0	0	$V^{11}_{(5/2)}$	$V^{01}_{(5)}$	$V^{11}_{(15/2)}$	$V^{01}_{(10)}$	$V^{11}_{(25/2)}$	$V^{01}_{(15)}$	$\cdots$
2	$\cdots$	0	0	0	$V^{10}_{(2)}$	$V^{00}_{(4)}$	$V^{10}_{(6)}$	$V^{00}_{(8)}$	$V^{10}_{(10)}$	$V^{00}_{(12)}$	$\cdots$
$\frac{3}{2}$	$\cdots$	0	0	0	$V^{11}_{(3/2)}$	$V^{01}_{(3)}$	$V^{11}_{(9/2)}$	$V^{01}_{(6)}$	$V^{11}_{(15/2)}$	$V^{01}_{(9)}$	$\cdots$
1	$\cdots$	0	0	0	$V^{10}_{(1)}$	$V^{00}_{(2)}$	$V^{10}_{(3)}$	$V^{00}_{(4)}$	$V^{10}_{(5)}$	$V^{00}_{(6)}$	$\cdots$
$\frac{1}{2}$	$\cdots$	0	$V^{11}_{(-1/2)}$	0	$V^{11}_{(1/2)}$	$V^{01}_{(1)}$	$V^{11}_{(3/2)}$	$V^{01}_{(2)}$	$V^{11}_{(5/2)}$	$V^{01}_{(3)}$	$\cdots$
0	$\cdots$	0	0	$\mathbf{R}^2$	0	0	0	0	0	0	$\cdots$
$-\frac{1}{2}$	$\cdots$	$V^{01}_{(1)}$	$V^{11}_{(1/2)}$	0	$V^{11}_{(-1/2)}$	0	0	0	0	0	$\cdots$
-1	$\cdots$	$V^{00}_{(2)}$	$V^{10}_{(1)}$	0	0	0	0	0	0	0	$\cdots$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Theorem 3.10 *The simple roots of the Baby Monster Lie algebra are the vectors $(1/2, n)$ with multiplicity $c^{10}(n)$ for integral n and with multiplicity $c^{11}(n)$ for non-integral n. The root $(\frac{1}{2}, -\frac{1}{2})$ is real, the others are imaginary. The primitive norm 0 vector $\rho = (-1/2, 0)$ is a Weyl vector.*

Proof. Our proof follows the proof of Theorem 7.2 in [Bor92]. Let $\mathbf{k}$ be the generalized Kac-Moody algebra with root lattice L', Cartan subalgebra $L' \otimes \mathbf{R}$ and simple roots as stated in the theorem. The denominator identity of the Lie algebra $\mathbf{k}$ is an identity in the multiplicative written group ring $\mathbf{C}[L']$. Writing $p^{1/2}$ and $q^{1/2}$ for the elements $e^{(1/2,0)}$ and $e^{(0,1/2)}$ one gets

$$e^{\rho} \prod_{\substack{m \in \frac{1}{2}\mathbf{Z},\, m>0 \\ n \in \frac{1}{2}\mathbf{Z}}} (1 - p^m q^n)^{u(m,n)} = \sum_{w \in W} \det(w) w \Big(e^{\rho} \sum_S (-1)^{|S|} e^{\sum_{\alpha \in S} \alpha} \Big)$$

where $u(m,n)$ is the multiplicity of the root $(m,n) \in L'$. The Weyl group W is isomorphic to $\mathbf{Z}_2$, with the nontrivial element exchanging $p^{1/2}$ and $q^{1/2}$, and the sum $\sum_S$ runs over all finite sets S of pairwise orthogonal imaginary simple roots. Since all imaginary simple roots have nontrivial scalar product with each other, one has $(-1)^{|S|} e^{\sum_{\alpha \in S} \alpha} = 1 - \sum_{n>0} c^{[1,n]}(n)\, pq^n$ and the right side gives

$$(p^{-1/2} - f(q) + q^{1/2}) - (q^{-1/2} - f(p) + p^{1/2}) = f(p) - f(q).$$

It follows now from Proposition 3.9 that the left side of the denominator identity of $\mathbf{k}$ is given by

$$p^{-1/2} \prod_{\substack{m \in \frac{1}{2}\mathbf{Z},\, m>0 \\ n \in \frac{1}{2}\mathbf{Z}}} (1 - p^m q^n)^{c^{[m,n]}(2mn)}.$$

Here, the exponents $c^{ij}(n)$ are the coefficients of the four modular functions $f^{00}(q)$, $f^{01}(q)$, $f^{10}(q)$ and $f^{11}(q)$.

Using Theorem 3.6, we see that $\mathbf{g}_{\mathbb{B}}^{\natural}$ and $\mathbf{k}$ have the same root multiplicities. The product in the denominator identity determines recursively the simple roots of $\mathbf{g}_{\mathbb{B}}^{\natural}$ because we have fixed a Cartan subalgebra and a fundamental Weyl chamber. It follows that $\mathbf{g}_{\mathbb{B}}^{\natural}$ and $\mathbf{k}$ have the same simple roots and are isomorphic. $\square$

Corollary 3.11 *The denominator identity of $\mathbf{g}_{\mathbb{B}}^{\natural}$ is*

$$p^{-1/2} \prod_{\substack{m \in \frac{1}{2}\mathbf{Z},\, m>0 \\ n \in \frac{1}{2}\mathbf{Z}}} (1 - p^m q^n)^{c^{[m,n]}(2mn)} = f(p) - f(q)$$

where $f(q) = f^{10}(q) + f^{11}(q) = q^{-1/2} + 95256\, q^{1/2} + \cdots$. and $c^{ij}(n)$ is the coefficient of q^n in $f^{ij}(q)$ and $[m,n]$ denotes the rest class of $(m,n) \in L'$ in $L'/L = A$.

Corollary 3.12 *The Baby Monster Lie algebra* $\mathbf{g}_{\mathbb{B}}^{\natural}$ *is isomorphic to the Lie algebra defined from the* t*-twisted denominator identity of the Monster Lie algebra* $\mathbf{g}$ *via generator and relations.*

Proof. A simple reorganization of the product in Corollary 3.11 transforms it to the form given in [Bor92], Sec. 10 for the t-twisted denominator identity of the Monster Lie algebra; cf. [Bor98], Example 13.9. □

4 The genus zero property

The Baby Monster Lie algebra is a L'-graded Lie algebra on which $2.\mathbb{B}$ acts by grading respecting automorphisms. We examine the equivariant denominator identity to conclude that the generalized Thompson series $T_{t,h}(q)$ are Hauptmoduls.

The functions arising in generalized Moonshine satisfy certain recursion formulas. For the case related to the Baby Monster, we will need the class of replicable functions.

Definition 4.1 (Replicable [Nor84]) Let

$$f(q) = q^{-1} + H_1\,q + H_2\,q^2 + H_3\,q^3 + \cdots$$

be a Laurent series with rational coefficients H_i, $i = 1,\,2,\,\ldots$. The bivarial transformation of f is defined as the series

$$\sum_{m,\,n\geq 1} H_{m,n}\,p^m q^n \quad = \quad -\log\Big(1 - pq\sum_{i=1}^{\infty} H_i\,\frac{p^i-q^i}{p-q}\Big)$$

so that the $H_{m,n}$ are polynomials in the H_i. The series f is called *replicable* if one has $H_{m,n} = H_{r,s}$ whenever $mn = rs$ and $(m,n) = (r,s)$ for the gcd's.

Remark: A function f as above is replicable if and only if for all positive integers n there exists a series $f^{(n)}(q) = q^{-1} + \sum_{i=1}^{\infty} H_i^{(n)}\,q^i$, called the n-th replicate of $f(q)$, such that

$$\sum_{\substack{a\,d=n\\ 0\leq b<d}} f^{(a)}(e^{2\pi ib/d}q^{a/d}) = P_n(f(q)) \qquad (6)$$

where $P_n(t)$ is the unique polynomial for which $P_n(f(q)) - q^{-n}$ is a series in positive powers of q; cf. [CN95], Lemma A.2. One has the relation (cf. [CN95], eq. (A.1))

$$H_{m,n} = \sum_{t|(m,n)} \frac{1}{t} H^{(t)}_{mn/t^2}. \qquad (7)$$

Replicable functions are determined by their first 25 coefficients. More precisely one has:

Proposition 4.2 (Norton [Nor84, CN95]) *If*

$$f = q^{-1} + H_1\, q + H_2\, q^2 + H_3\, q^3 + \cdots$$

is replicable, then f is determined by the values of H_1, H_2, H_3, H_4, H_5, H_7, H_8, H_9, H_{11}, H_{17}, H_{19} and H_{23}. □

The idea of the proof is that $H_{m,n}$ is a polynomial in $H_1, \ldots, H_{m+n+1}$ which leads to recursion formulas for H_{m+n+1} in terms of $H_1, \ldots, H_{r+s+1}$ whenever one finds a pair (r, s) with $r \cdot s = m \cdot n$, $\gcd(m, n) = \gcd(r, s)$ and $r + s < m + n$.

As with any generalized Kac-Moody algebra, we can decompose the Baby Monster Lie algebra into a direct sum $E \oplus H \oplus F$ of subalgebras, where H is the Cartan subalgebra and E and F are the subalgebras generated by the positive and negative root spaces. There is the equality

$$\Lambda(E) \;=\; \sum_{i\geq 0}(-1)^i\Lambda^i(E) \;=\; \sum_{i\geq 0}(-1)^i H_i(E) \;=\; H(E) \tag{8}$$

of virtual L'-graded modules for $2.\mathbb{B}$.

Proposition 4.3 *Let $p^{1/2}$ and $q^{1/2}$ stand for one dimensional vector spaces of degrees $(1/2, 0)$ and $(0, 1/2)$ in L' and let $V^{ij}_{(n)}$ have the degree $0 \in L'$ for all $n \in \frac{1}{2}\mathbf{Z}$ and $(i, j) \in A$. Then the homology $H_*(E)$ of the Baby Monster Lie algebra are the following L'-graded vector spaces:*

$$\begin{aligned}
H_0(E) &= V^{11}_{-1/2} \cong \mathbf{R} \textit{ with character } 1,\\
H_1(E) &= \bigoplus_{n\in\frac{1}{2}\mathbf{Z}} V^{[1/2,n]}_{(n)}\, p^{1/2} q^{n/2} \textit{ with character } p^{1/2}(f^{10}(q) + f^{11}(q)),\\
H_2(E) &= p^{1/2} \bigoplus_{\substack{m\in\frac{1}{2}\mathbf{Z}\\ m>0}} V^{[m,1/2]}_{(m)}\, p^{m/2} \textit{ with character } p^{1/2}(f^{01}(p) + f^{11}(p) - p^{-1/2}),\\
H_i(E) &= 0 \quad \textit{for } i \geq 3.
\end{aligned}$$

Proof. The proof is nearly identical to the case of the Monster Lie algebra studied in [Bor92], Sec. 8. The Weyl vector ρ of the Baby Monster Lie algebra is the norm 0 vector $(-1/2, 0)$. The positive roots are the roots of degree (m, n) with $m \geq 0$. We use from [Bor92] that the homology space $H_i(E)$ of a generalized Kac-Moody Lie algebras can be identified with the subspace of elements of $\Lambda^i(E)$ whose degree r in the root lattice satisfies $(r, r + 2\rho) = 0$. For $H_i(E)$, we have therefor to find the components of degree (m, n) in $\Lambda^i(E)$ with $2m \cdot n = n$. The case $H_0(E)$ is clear because $\Lambda^0(E) = \mathbf{R}$ sitting in degree $(0, 0)$. The others can be easily read of from Table 1. $H_1(E)$ is the subspace of $\Lambda^1(E) = E$ of degree $(m, 0)$ or $(1/2, n)$. Only the second type contributes. For $i \geq 2$ there are no elements in $\Lambda^i(E)$ of degree $(1/2, n)$ since for all homogenous elements of degree (m,n) in E one has $m \geq 1/2$. The one dimensional space with degree $(1/,-1/2)$ belonging to the real positive simple root is the only component of degree (m,n) with $n \leq 0$. So the only possibility for an element $\Lambda^2(E)$ to have degree $(m,0)$ is for it to be the exterior product of two elements of E of degrees $(1/2,-1/2)$ and $(m-1/2,1/2)$. Therefore, $H_2(E)$ has to be the sum of pieces of degrees $(m,0)$ for $m \geq 2$ isomorphic to $V_{(m-1/2)}^{[m-1/2,1/2]}$. There are no elements of degree $(m,0)$ in $\Lambda^i(E)$ for $i \geq 3$ and so the homology groups $H_i(E)$ vanish for $i \geq 3$. $\square$

Substituting the Homology groups described in Proposition 4.3 into equation (8) we obtain

$$p^{-1/2}\Lambda\Bigg(\sum_{\substack{m\in\frac{1}{2}\mathbf{Z},\, m>0\\ n\in\frac{1}{2}\mathbf{Z}}} V_{(2mn)}^{[m,n]}\, p^m q^n\Bigg) = \sum_{m\in\frac{1}{2}\mathbf{Z}} V_{(m)}^{[m,1]}\, p^m - \sum_{n\in\frac{1}{2}\mathbf{Z}} V_{(n)}^{[1,n]}\, q^n. \tag{9}$$

Theorem 4.4 *The Baby Monster Thompson series $T_{t,h}(q^2)$ for $h \in 2.\mathbb{B}$ are replicable functions.*

Proof. We recall the elementary fact from character theory that for a finite dimensional vector space U acted on by a finite group G there is the identity $\mathrm{Tr}(g \mid \Lambda(U)) = \exp(-\sum_{i>0}\mathrm{Tr}(g^i \mid U)/i)$. The same identity is well-defined and true for the infinite dimensional vector space E graded by L' since all its pieces of fixed degree r are finite dimensional and the expansion on both sides contains only a finite number of terms in each degree. (If U_r has degree $r \in L'$ then $\mathrm{Tr}(g^i \mid U)$ is understood to have degree $i \cdot r$.) Taking the trace of an element $h \in 2.\mathbb{B}$ on both sides of (9) leads to the identity

$$\begin{aligned} p^{-1/2}\exp\Bigg(-\sum_{i>0}\sum_{\substack{m\in\frac{1}{2}\mathbf{Z},\ m>0\\ n\in\frac{1}{2}\mathbf{Z}}} \mathrm{Tr}(h^i \mid V_{(2mn)}^{[m,n]})\, p^{im}q^{in}/i\Bigg) \\ = \sum_{m\in\frac{1}{2}\mathbf{Z}} \mathrm{Tr}(h \mid V_{(m)}^{[m,1]})\, p^m - \sum_{n\in\frac{1}{2}\mathbf{Z}} \mathrm{Tr}(h \mid V_{(n)}^{[1,n]})\, q^n. \end{aligned} \tag{10}$$

Replacing $p^{1/2}$ by p and $q^{1/2}$ by q and rearranging gives

$$\sum_{m,n>0}\sum_{t\mid(m,n)} \frac{1}{t}\cdot\mathrm{Tr}(h^t \mid V_{(mn/(2t^2))}^{[\frac{m}{t},\frac{n}{t}]'})\, p^m q^n = \log(p^{-1}-q^{-1}) - \log(T_{t,h}(p^2)-T_{t,h}(q^2)),$$

where $[\ ,\ ]' : \mathbf{Z}\times\mathbf{Z} \longrightarrow \mathbf{Z}_2\times\mathbf{Z}_2$ is now just the reduction modulo 2. Using Definition 4.1, the last equation shows that the coefficients of the bivariate transform of $T_{t,h}(q^2)$ can be written as

$$H_{m,n} = \sum_{t\mid(m,n)} \frac{1}{t}\cdot \mathrm{Tr}(h^t \mid V_{(mn/(2t^2))}^{[\frac{m}{t},\frac{n}{t}]'}) \tag{11}$$

We have to show that $H_{m,n} = H_{m',n'}$ for all pairs (m',n') with $m'n' = mn$ and $(m,n) = (m',n')$. We write $m = 2^\alpha \tilde{m}$ and $n = 2^\beta \tilde{n}$ for natural numbers $\tilde{m}$ and $\tilde{n}$ with $(2,\tilde{m}) = (2,\tilde{n}) = 1$ and similar $t = 2^\gamma \tilde{t}$ with $(2,\tilde{t}) = 1$. The term

$$\mathrm{Tr}(h^t \mid V^{[\frac{m}{t},\frac{n}{t}]'}_{(mn/(2t^2))}) = \mathrm{Tr}(h^t \mid V^{[2^{\alpha-\gamma},2^{\beta-\gamma}]'}_{(mn/(2t^2))})$$

depends only on mn since if we replace (m,n) by (m',n') then either $\alpha' = \alpha$ and $\beta' = \beta$ or $\alpha' = \beta$ and $\beta' = \alpha$. In the second case we have $\mathrm{Tr}(h^t \mid V^{[2^{\alpha-\gamma},2^{\beta-\gamma}]'}) = \mathrm{Tr}(h^t \mid V^{[2^{\beta-\gamma},2^{\alpha-\gamma}]'})$ since $V^{10} = V^{01}$ as $2.\mathbb{B}$ modules. □

Remark: The Thompson series $T_g(q)$ for the Moonshine module belong to the smaller class of *completely replicable functions.* These are replicable functions f for which all the replicates $f^{(s)}$ are themself replicable. A completely replicable function f is determined by the coefficients $H_1^{(s)}$, $H_2^{(s)}$, $H_3^{(s)}$ and $H_5^{(s)}$ of all replicates $f^{(s)}$. For the Thompson series $T_g(q)$ one has $T_g^{(s)}(q) = T_{g^s}(q)$ and therefore all the $T_g(q)$ are determined from the knowledge of the five Monster representations $V_2^\natural$, $V_3^\natural$, $V_4^\natural$, $V_5^\natural$ and $V_6^\natural$.

The reason that the function $T_{t,h}(q^2)$ is not completely replicable in general is that its replications $T_{t,h}^{(s)}(q^2)$ fails to be the same as the replicable functions $\sum_{n=-1}^\infty \mathrm{Tr}(h^s \mid V_{2n}^{[1,n]})\, q^n$:

Let $\widetilde{H}_{m,n}^{(s)}$ be the coefficients of the bivariate transform of

$$\sum_{n=-1}^{\infty} \mathrm{Tr}(g^s \mid V^{[1,n]}_{(n/2)})\, q^n.$$

Then (replacing h by h^s in equation (11))

$$\widetilde{H}_{m,n}^{(s)} = \sum_{t|(m,n)} \frac{1}{t} \cdot \mathrm{Tr}(h^{ts} \mid V^{[(\frac{m}{t},\frac{n}{t}]'}_{(mn/2)}).$$

For the coefficients of the the bivariate transform the s-th replication of $T_{t,h}(q^2)$ one has by (7) on the other hand:

$$H_{m,n}^{(s)} = \sum_{t|(m,n)} \frac{1}{t} \cdot \mathrm{Tr}(g^{ts} \mid V^{[1,\frac{n}{t}]'}_{(mn/2)}).$$

But in general $V^{[1,\frac{n}{t}]'}_{(mn/2)} \neq V^{[\frac{m}{t},\frac{n}{t}]'}_{(mn/2)}$.

It has been proven by Cummins and Norton [CN95], that normalized Hauptmoduls with rational coefficients for a group $\Gamma \subset \mathrm{PGL}_2(\mathbf{Q})^+$ of genus zero, containing a $\Gamma_0(N)$ with finite index and $\tau \mapsto \tau + k$ precisely when k is an integer are replicable functions. It is a conjecture of Norton [Nor84] that all normalized replicable functions with rational coefficients are Hauptmoduls for such genus zero modular groups or are of the trivial form $q^{-1} + aq$. A list of 619 such genus

Table 2: The names of the Thompson series associated to the conjugacy classes of $2.\mathbb{B}$ (quite useless)

$2A$, $4\tilde{}b$, $2a$, $4A$, $2B$, $8\tilde{}b$, $4C$, $6A$, $12\tilde{}d$, $6D$, $12\tilde{}f$, $4a$, $4B$, $4C$, $8a$, $8A$, $8\tilde{}d$, $4D$, $8B$, $8E$, $16\tilde{}a$, $8D$, $10A$, $20\tilde{}c$, $10C$, $20\tilde{}d$, $6a$, $6b$, $12A$, $6C$, $6c$, $12c$, $24\tilde{}f$, $12B$, $6E$, $12E$, $12H$, $12\tilde{}h$, $12I$, $24\tilde{}h$, $14A$, $28\tilde{}c$, $8a$, $8b$, $8c$, $8B$, $8C$, $8D$, $16\tilde{}d$, $8E$, $8F$, $16A$, $16C$, $16\tilde{}e$, $16a$, $16B$, $16d$, $18A$, $36\tilde{}h$, $18B$, $36\tilde{}e$, $10a$, $20A$, $10B$, $40\tilde{}c$, $20C$, $10E$, $20d$, $20F$, $20\tilde{}g$, $22A$, $44\tilde{}b$, $12a$, $12G$, $12b$, $12C$, $24a$, $24\tilde{}j$, $24\tilde{}k$, $12E$, $24b$, $12d$, $24B$, $24\tilde{}m$, $24c$, $24A$, $12I$, $24C$, $24\tilde{}o$, $12F$, $24h$, $24\tilde{}q$, $24H$, $24I$, $24\tilde{}r$, $48\tilde{}c$, $26A$, $52\tilde{}c$, $14a$, $14c$, $56\tilde{}d$, $28B$, $14B$, $28C$, $30B$, $60\tilde{}c$, $30F$, $60\tilde{}l$, $16b$, $16c$, $16B$, $16A$, $16a$, $16A$, $32B$, $32A$, $32\tilde{}e$, $34A$, $68\tilde{}b$, $18c$, $18c$, $36A$, $18C$, $36f$, $36D$, $36\tilde{}q$, $72\tilde{}c$, $38A$, $76\tilde{}b$, $20a$, $20c$, $20b$, $20B$, $40a$, $40B$, $40\tilde{}h$, $40\tilde{}i$, $80\tilde{}a$, $20E$, $40C$, $42A$, $84\tilde{}e$, $22a$, $22a$, $44A$, $22B$, $46C$, $92\tilde{}b$, $46C$, $92\tilde{}b$, $24d$, $24e$, $24g$, $24f$, $24b$, $24c$, $24A$, $48\tilde{}h$, $48\tilde{}i$, $48A$, $24H$, $48\tilde{}j$, $48\tilde{}k$, $24F$, $48h$, $50A$, $100\tilde{}c$, $104\tilde{}c$, $26a$, $54A$, $108\tilde{}g$, $28A$, $28C$, $28a$, $56a$, $56A$, $56\tilde{}g$, $30a$, $30d$, $120\tilde{}d$, $60B$, $30C$, $60a$, $60D$, $60\tilde{}m$, $60C$, $30G$, $60C$, $30G$, $62A$, $124\tilde{}b$, $62A$, $124\tilde{}b$, $32B$, $32B$, $32b$, $32b$, $66A$, $132\tilde{}c$, $136\tilde{}b$, $34a$, $34a$, $70A$, $140\tilde{}b$, $36C$, $72a$, $72\tilde{}p$, $72\tilde{}q$, $38a$, $38a$, $78A$, $156\tilde{}d$, $40b$, $40c$, $40A$, $80a$, $80\tilde{}e$, $80\tilde{}e$, $42a$, $42b$, $168\tilde{}c$, $44c$, $44c$, $92A$, $46A$, $92A$, $46A$, $94A$, $188\tilde{}b$, $94A$, $188\tilde{}b$, $48a$, $48b$, $208\tilde{}a$, $208\tilde{}a$, $110A$, $220\tilde{}b$, $56a$, $56a$, $60b$, $120a$, $120\tilde{}g$, $120\tilde{}h$, $66a$, $66a$, $70a$, $70a$

The conjugacy classes are listed in the order of the atlas of finite groups [CCN$^+$85], the one or two lifts of a class in $\mathbb{B}$ to $2.\mathbb{B}$ are grouped together. The number N in the names "NX", "Nx" or "$N\tilde{}x$" is the order of an element in this class. "NX" denotes the monstrous moonshine functions as listed in [CN79], "Nx" denotes the remaining completely replicable functions as listed in [ACMS92], "$N\tilde{}x$" denote the remaining replicable functions as listed in [Nor01].

Table 3: The first 25 Head characters of $V^\natural(t)$ for the group $2.\mathbb{B}$

H_{-1}: χ_1, | H_0: 0 | H_1: $\chi_1+\chi_2$ | H_2: χ_{185} | H_3: $2\chi_1+\chi_2+\chi_3+\chi_4$

H_4: $2\chi_{185}+\chi_{186}$ | H_5: $3\chi_1+3\chi_2+2\chi_3+\chi_4+\chi_6+\chi_7$ | H_6: $4\chi_{185}+2\chi_{186}+\chi_{187}$

H_7: $6\chi_1+5\chi_2+4\chi_3+3\chi_4+\chi_5+2\chi_6+\chi_7+\chi_8+\chi_9+\chi_{10}$

H_8: $8\chi_{185}+4\chi_{186}+3\chi_{187}+\chi_{188}$

H_9: $8\chi_1+10\chi_2+7\chi_3+5\chi_4+2\chi_5+5\chi_6+4\chi_7+2\chi_8+2\chi_9+\chi_{10}+\chi_{11}+\chi_{12}+\chi_{14}+\chi_{18}$

H_{10}: $14\chi_{185}+9\chi_{186}+7\chi_{187}+3\chi_{188}+\chi_{189}+\chi_{192}$

H_{11}: $14\chi_1+16\chi_2+14\chi_3+11\chi_4+5\chi_5+9\chi_6+7\chi_7+5\chi_8+5\chi_9+4\chi_{10}+3\chi_{11}+2\chi_{12}+\chi_{13}+3\chi_{14}+\chi_{16}+2\chi_{18}+\chi_{20}+\chi_{23}+\chi_{27}$

H_{12}: $25\chi_{185}+17\chi_{186}+16\chi_{187}+8\chi_{188}+3\chi_{189}+3\chi_{192}+\chi_{193}+\chi_{194}$

H_{13}: $20\chi_1+29\chi_2+23\chi_3+18\chi_4+9\chi_5+19\chi_6+16\chi_7+11\chi_8+11\chi_9+8\chi_{10}+6\chi_{11}+6\chi_{12}+2\chi_{13}+8\chi_{14}+2\chi_{16}+\chi_{17}+6\chi_{18}+2\chi_{20}+3\chi_{23}+\chi_{24}+2\chi_{27}+2\chi_{29}+\chi_{31}+\chi_{32}+\chi_{34}$

H_{14}: $43\chi_{185}+32\chi_{186}+33\chi_{187}+19\chi_{188}+8\chi_{189}+8\chi_{192}+4\chi_{193}+4\chi_{194}+\chi_{195}+\chi_{196}$

H_{15}: $32\chi_1+45\chi_2+41\chi_3+34\chi_4+19\chi_5+34\chi_6+29\chi_7+23\chi_8+23\chi_9+19\chi_{10}+14\chi_{11}+12\chi_{12}+6\chi_{13}+17\chi_{14}+5\chi_{16}+\chi_{17}+14\chi_{18}+6\chi_{20}+9\chi_{23}+4\chi_{24}+7\chi_{27}+\chi_{28}+5\chi_{29}+\chi_{30}+3\chi_{31}+3\chi_{32}+\chi_{33}+2\chi_{34}+\chi_{38}+\chi_{41}+\chi_{42}+\chi_{43}+\chi_{50}+\chi_{51}$

H_{16}: $72\chi_{185}+59\chi_{186}+65\chi_{187}+41\chi_{188}+20\chi_{189}+20\chi_{192}+11\chi_{193}+12\chi_{194}+4\chi_{195}+4\chi_{196}+\chi_{197}+2\chi_{198}+\chi_{204}$

H_{17}: $46\chi_1+76\chi_2+68\chi_3+56\chi_4+33\chi_5+63\chi_6+58\chi_7+44\chi_8+45\chi_9+35\chi_{10}+29\chi_{11}+27\chi_{12}+12\chi_{13}+38\chi_{14}+11\chi_{16}+5\chi_{17}+32\chi_{18}+13\chi_{20}+21\chi_{23}+11\chi_{24}+17\chi_{27}+2\chi_{28}+14\chi_{29}+3\chi_{30}+10\chi_{31}+9\chi_{32}+3\chi_{33}+8\chi_{34}+5\chi_{38}+4\chi_{41}+3\chi_{42}+2\chi_{43}+4\chi_{50}+3\chi_{51}+\chi_{54}+\chi_{56}+\chi_{61}+\chi_{63}+\chi_{64}+\chi_{69}+\chi_{70}$

H_{18}: $119\chi_{185}+104\chi_{186}+124\chi_{187}+85\chi_{188}+44\chi_{189}+45\chi_{192}+29\chi_{193}+32\chi_{194}+12\chi_{195}+13\chi_{196}+4\chi_{197}+8\chi_{198}+\chi_{202}+\chi_{203}+4\chi_{204}+\chi_{209}+2\chi_{216}$

H_{19}: $71\chi_1+118\chi_2+114\chi_3+99\chi_4+63\chi_5+109\chi_6+101\chi_7+84\chi_8+86\chi_9+72\chi_{10}+58\chi_{11}+52\chi_{12}+28\chi_{13}+75\chi_{14}+25\chi_{16}+9\chi_{17}+67\chi_{18}+31\chi_{20}+50\chi_{23}+27\chi_{24}+41\chi_{27}+6\chi_{28}+33\chi_{29}+10\chi_{30}+24\chi_{31}+23\chi_{32}+9\chi_{33}+19\chi_{34}+14\chi_{38}+13\chi_{41}+9\chi_{42}+9\chi_{43}+\chi_{48}+12\chi_{50}+12\chi_{51}+3\chi_{54}+5\chi_{56}+2\chi_{60}+4\chi_{61}+3\chi_{63}+4\chi_{64}+4\chi_{69}+3\chi_{70}+\chi_{72}+\chi_{73}+2\chi_{74}+\chi_{75}+2\chi_{76}+2\chi_{77}+\chi_{94}$

H_{20}: $194\chi_{185}+180\chi_{186}+229\chi_{187}+169\chi_{188}+93\chi_{189}+96\chi_{192}+69\chi_{193}+77\chi_{194}+33\chi_{195}+37\chi_{196}+12\chi_{197}+25\chi_{198}+\chi_{201}+4\chi_{202}+5\chi_{203}+14\chi_{204}+\chi_{208}+4\chi_{209}+10\chi_{216}+\chi_{219}+\chi_{220}+2\chi_{221}$

H_{21}: $101\chi_1+190\chi_2+183\chi_3+161\chi_4+107\chi_5+191\chi_6+186\chi_7+154\chi_8+159\chi_9+130\chi_{10}+111\chi_{11}+104\chi_{12}+54\chi_{13}+149\chi_{14}+49\chi_{16}+23\chi_{17}+138\chi_{18}+64\chi_{20}+106\chi_{23}+61\chi_{24}+89\chi_{27}+13\chi_{28}+77\chi_{29}+25\chi_{30}+59\chi_{31}+56\chi_{32}+23\chi_{33}+51\chi_{34}+39\chi_{38}+34\chi_{41}+23\chi_{42}+21\chi_{43}+2\chi_{48}+33\chi_{50}+30\chi_{51}+11\chi_{54}+17\chi_{56}+6\chi_{60}+13\chi_{61}+12\chi_{63}+16\chi_{64}+2\chi_{67}+14\chi_{69}+13\chi_{70}+2\chi_{71}+3\chi_{72}+3\chi_{73}+7\chi_{74}+5\chi_{75}+8\chi_{76}+8\chi_{77}+3\chi_{87}+\chi_{89}+2\chi_{92}+4\chi_{94}+\chi_{96}+\chi_{100}+3\chi_{102}+\chi_{105}+\chi_{112}+\chi_{114}$

H_{22}: $311\chi_{185}+308\chi_{186}+412\chi_{187}+325\chi_{188}+190\chi_{189}+198\chi_{192}+153\chi_{193}+174\chi_{194}+82\chi_{195}+94\chi_{196}+34\chi_{197}+70\chi_{198}+3\chi_{201}+12\chi_{202}+18\chi_{203}+42\chi_{204}+4\chi_{208}+14\chi_{209}+36\chi_{216}+5\chi_{219}+6\chi_{220}+12\chi_{221}+2\chi_{226}+2\chi_{227}+\chi_{231}+\chi_{237}$

H_{23}: $153\chi_1+291\chi_2+298\chi_3+270\chi_4+190\chi_5+321\chi_6+318\chi_7+278\chi_8+288\chi_9+246\chi_{10}+210\chi_{11}+192\chi_{12}+111\chi_{13}+279\chi_{14}+\chi_{15}+100\chi_{16}+44\chi_{17}+269\chi_{18}+135\chi_{20}+222\chi_{23}+133\chi_{24}+191\chi_{27}+33\chi_{28}+164\chi_{29}+61\chi_{30}+128\chi_{31}+126\chi_{32}+57\chi_{33}+114\chi_{34}+94\chi_{38}+3\chi_{40}+87\chi_{41}+58\chi_{42}+59\chi_{43}+2\chi_{44}+7\chi_{48}+\chi_{49}+83\chi_{50}+80\chi_{51}+28\chi_{54}+47\chi_{56}+2\chi_{57}+20\chi_{60}+38\chi_{61}+32\chi_{63}+45\chi_{64}+8\chi_{67}+3\chi_{68}+42\chi_{69}+38\chi_{70}+6\chi_{71}+10\chi_{72}+12\chi_{73}+24\chi_{74}+20\chi_{75}+29\chi_{76}+28\chi_{77}+3\chi_{82}+2\chi_{83}+2\chi_{85}+13\chi_{87}+2\chi_{88}+5\chi_{89}+8\chi_{92}+3\chi_{93}+17\chi_{94}+5\chi_{96}+2\chi_{98}+6\chi_{100}+12\chi_{102}+3\chi_{105}+3\chi_{109}+4\chi_{112}+5\chi_{114}+2\chi_{116}+2\chi_{117}+2\chi_{121}+\chi_{124}+\chi_{125}+3\chi_{127}+\chi_{139}+\chi_{142}+\chi_{147}+2\chi_{149}$

zero modular groups, together with the first 25 coefficients of the corresponding replicable normalized Hauptmoduls has been obtained by Norton [Nor01] and it is a conjecture that this list contains all replicable functions with integral coefficients. In case someone proves this conjecture (as it has been done for completely replicable functions by Cummins and Gannon [CG97]), the proof of our main Theorem is finished. Since this seems not to be the case yet, we have to identify all functions individually.

Theorem 4.5 *The first 25 coefficients of the generalized Thompson series $T_{t,h}(q^2)$ for $h \in 2.\mathbb{B}$ are the generalized Moonshine functions conjectured by Norton and listed in Table 2.*

Proof. Borcherds proof of the Moonshine conjecture for the Moonshine module makes it possible to determine the decomposition of the Monster representations $V_n^\natural$ into irreducible constituents. For the range $0 \leq n \leq 13$ which we need, they can be found in the table in [MS90]. By Lemma 1, we have decompositions

$$\begin{aligned} V^{00} &= VB^\natural(0) \otimes L(0) \oplus VB^\natural(1) \otimes L(1), \\ V^{01} = V^{10} &= VB^\natural(2) \otimes L(2), \\ V^{11} &= VB^\natural(0) \otimes L(1) \oplus VB^\natural(1) \otimes L(0) \end{aligned}$$

of $VB^\natural(0) \otimes L(0)$-modules compatible with the action of $2.\mathbb{B}$. Using $V^\natural = V^{00} \oplus V^{01}$ and restricting the Monster representations $V_n^\natural$ to $2.\mathbb{B}$ allows us to determine the decomposition of $V_n^{00} \oplus V_n^{01}$, $0 \leq n \leq 13$ into $2.\mathbb{B}$-irreducibles by using the character tables of the Monster [ea] and the Baby Monster and its two-fold cover [Hun]. Since the central element t of $2.\mathbb{B}$ acts on V^{00} resp. V^{01} by multiplication with 1 resp. -1, we can decide if an irreducible constituent belongs to V_n^{00} or V_n^{01}. The result for $V_n^{01} \cong V_n^{10}$ in the range $0 \leq n \leq 12$ can be found in Table 3 (with V_i^{10} corresponding to the Head character H_{2i-2}).

The computation of the decomposition of V_n^{11}, $n = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{25}{2}$ into $2.\mathbb{B}$-irreducibles is more involved. First, we deduce from the $\mathbb{B}$-decomposition of V_n^{00}, $0 \leq n \leq 5$ the $\mathbb{B}$-decomposition of shorter Moonshine module components $VB^\natural(0)_n$, $n \leq 5$ and $VB^\natural(1)_n$, $n = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{9}{2}$. This is possible since there is only one way to combine the $\mathbb{B}$-irreducibles to get the conformal characters of $VB^\natural(0)$ and $VB^\natural(1)$. We also use that $VB^\natural(0)_n$ and $VB^\natural(1)_n$ are unitary modules of the Virasoro algebra of central charge $c > 1$; cf. [Höh95], Sect. 5.1 and [HL94]. The result is shown in [Höh95], Lemma 4.2.10. From this we obtain the decompositions of V_n^{11}, $n = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{9}{2}$. (corresponding to the Head characters H_i, $i = -1, 1, 3, 5$ and 7 in Table 3). Next, we expand the logarithm of equation (10) up to the 4-th power in p and compare coefficients. This leads to the following recursion formulas for the coefficients $a_h(i)$ of the Thompson series $T_{t,h}(q^2) = \sum_{i=-1}^{\infty} a_h(i)\, q^i$:

$$\begin{aligned} a_h(9) &= a_h(2)^2 + a_h(1)^3/3 + a_h(1)a_h(3) + a_h(5) - a_{h^3}(1)/3, \\ a_h(6) &= a_h(4) + a_h(2)a_h(1), \end{aligned}$$

$$
\begin{aligned}
a_h(10) &= a_h(6)+a_h(4)a_h(1)+a_h(2)a_h(3),\\
a_h(12) &= a_h(6)+a_h(4)a_h(1)+a_h(2)a_h(1)^2+2a_h(2)a_h(3),\\
a_h(14) &= a_h(8)+a_h(6)a_h(1)+a_h(4)a_h(3)+a_h(2)a_h(5),\\
a_h(36) &= 3a_h(2)a_h(4)^2+3a_h(2)^2a_h(6)+a_h(12)+3a_h(2)^2a_h(4)a_h(1)\\
&\quad +a_h(10)a_h(1)+a_h(8)a_h(1)^2+a_h(6)a_h(1)^3+a_h(2)^3a_h(3)\\
&\quad +3a_h(8)a_h(3)+4a_h(6)a_h(1)a_h(3)+3a_h(4)a_h(1)^2a_h(3)\\
&\quad +3a_h(4)a_h(3)^2+3a_h(2)a_h(1)a_h(3)^2+3a_h(6)a_h(5)\\
&\quad +4a_h(4)a_h(1)a_h(5)+3a_h(2)a_h(1)^2a_h(5)+6a_h(2)a_h(3)a_h(5)\\
&\quad +3a_h(4)a_h(7)+4a_h(2)a_h(1)a_h(7)+2a_h(2)a_h(9),\\
a_h(11) &= \big(-a_h(4)^3/3-2a_h(2)a_h(4)a_h(6)-a_h(2)^2a_h(8)-a_h(14)+a_h(36)\\
&\quad +a_{h^3}(4)/3-a_h(12)a_h(1)-a_h(10)a_h(1)^2-2a_h(10)a_h(3)\\
&\quad -2a_h(8)a_h(1)a_h(3)-a_h(6)a_h(3)^2-2a_h(8)a_h(5)\\
&\quad -2a_h(6)a_h(1)a_h(5)-2a_h(4)a_h(3)a_h(5)-a_h(2)a_h(5)^2-2a_h(6)a_h(7)\\
&\quad -2a_h(4)a_h(1)a_h(7)-2a_h(2)a_h(3)a_h(7)\\
&\quad -2a_h(4)a_h(9)-2a_h(2)a_h(1)a_h(9)\big)/(2a_h(2)).
\end{aligned}
$$

Therfore, $a_h(9)$ is determined by $a_h(i)$, $a_{h^3}(i)$ for $0 \leq i \leq 5$ and $i = 7$. The coefficient $a_h(11)$ is determined by these coefficients provided $a_h(2) = \chi_{185}(h)$ is different from zero. We get enough equations to determine H_{11} completely from the $\mathbb{B}$-decomposition of $V_n^\natural$. In the next step we expand the logarithm of equation (10) up to the 6-th power in p. This leads to more complicated recursion formulas for $a_h(17)$, $a_h(19)$ and $a_h(23)$ in terms of $a_{h^k}(i)$ for $0 \leq i \leq 5$ and $i = 7$, 11 and for all powers h^k. provided $a_h(1) = \chi_2(h)$ is different from zero. There are 200 such conjugacy classes and the resulting equations, again using the $\mathbb{B}$-decomposition of the $V_n^\natural$, are sufficient to finally determine H_{17}, H_{19} and H_{23}. The remaining H_i are determined using Proposition 4.2. The result is shown in Table 3. Comparing the $a_g(i)$, $i \leq 23$ with the list in [Nor01] we obtain Table 2. □

Theorem 4.4, Theorem 4.5 and Proposition 4.2 provide a proof of our main Theorem:

Theorem 4.6 (generalized Moonshine for the Baby Monster) *The generalized Thompson series $T_{t,h}(q^2)$ for the t-twisted module $V^\natural(t)$ of the Moonshine module are Hauptmoduls for genus zero modular groups $\Gamma_{t,h}$ of $\mathrm{PGL}_2(\mathbf{Q})$ containing some $\Gamma_0(N)$ as normal subgroups.* □

Corollary 4.7 (shorter Moonshine module) *The Thompson series for the shorter Moonshine module for an element $(\pm 1, h) \in \mathbf{Z}_2 \times \mathbb{B} = \mathrm{Aut}(VB^\natural)$ are*

$$
\begin{aligned}
\mathrm{Tr}((1,h)\mid VB^\natural) &= \frac{1}{2}\big(\mathrm{Tr}(h\mid V^{00})+\mathrm{Tr}(h\mid V^{11})\big)\big/\big(\chi_{L_{1/2}(0)}+\chi_{L_{1/2}(1)}\big)\\
\mathrm{Tr}((-1,h)\mid VB^\natural) &= \frac{1}{2}\big(\mathrm{Tr}(h\mid V^{00})-\mathrm{Tr}(h\mid V^{11})\big)\big/\big(\chi_{L_{1/2}(0)}-\chi_{L_{1/2}(1)}\big)
\end{aligned}
$$

where

$$\begin{aligned}\mathrm{Tr}(h \mid V^{00}) &= \frac{1}{2}\Big(\mathrm{Tr}(\tilde{h} \mid V^\natural) + \mathrm{Tr}(\tilde{h}' \mid V^\natural)\Big) = \frac{1}{2}(T_{\tilde{h}}(q) + T_{\tilde{h}'}(q)) \\ \mathrm{Tr}(h \mid V^{11}) &= \frac{1}{2}\Big(\mathrm{Tr}(\tilde{h} \mid V^\natural(t)) + \mathrm{Tr}(\tilde{h}' \mid V^\natural(t))\Big) = \frac{1}{2}(T_{t,\tilde{h}}(q) + T_{t,\tilde{h}'}(q))\end{aligned}$$

and $\tilde{h}$ *and* $\tilde{h}'$ *denote the two lifts of* $h \in \mathbb{B}$ *to* $2.\mathbb{B}$.

Proof. Clear from decomposition 1 □

Matsuo defined in [Mat01] a class $\mathcal{S}^n$ of vertex operators algebras. A vertex operator algebra V with $V_0 = \mathbf{C1}$ and $V_1 = 0$ is of class $\mathcal{S}^n$ if the fixpoints of the action of $\mathrm{Aut}\, V$ on V_k for $k \leq n$ belong to the vertex operator subalgebra V_ω generated by the Virasoro element ω. It is clear that if V is of class $\mathcal{S}^n$, it is also of class $\mathcal{S}^m$ for $m < n$. Vertex operators algebras of class $\mathcal{S}^n$ have to satisfy certain trace identities. For vertex operators algebras with a real form and positive definite invariant bilinear form which are in class $\mathcal{S}^6$ only finitely many pairs (d, c), where d is the dimension of V_2 and c the central charge, are allowed (cf. [Mat01], Table 3.1, for c a half-integer). One example is the Moonshine module $V^\natural$ which is even of class $\mathcal{S}^{11}$. The even part $VB^\natural(0)$ of the shorter Moonshine module is a candidate for the class $\mathcal{S}^6$.

Corollary 4.8 *The even part* $VB^\natural(0)$ *of the shorter Moonshine module is a vertex operator algebra of class* $\mathcal{S}^7$.

Proof. The multiplicity of the trivial $\mathbb{B}$-representation in $VB^\natural(0)_n$, $0 \leq n \leq 7$, as determined from Corollary 4.7 coincide with the dimensions of $(V_\omega)_n$. □

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