Differential Topology — Spring 2014

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Final Exam   May 11, 2014

Please return your solutions until Friday, May 16, 5:00 p.m. to my mailbox.

Problem 1: Show that if $\mathbf{R}^n$ with its usual topology is also regarded as a group under vector addition, then the quotient group $\mathbf{R}^n/\mathbf{Z}^n$ is an n-dimensional differentiable manifold when given the quotient topology.

Problem 2: Let y be a regular value of a differentiable map $f : X \longrightarrow Y$ where X is compact and X and Y are differentiable manifolds of the same dimension. Show that there exists an open neighborhood U of y such that $f^{-1}(U)$ is a disjoint union of finitely many open sets V_i, $i = 1, \ldots, n$ and, for each i, $f|_{V_i}$ is a diffeomorphism with U.

Problem 3: Let M be a compact n-dimensional differentiable manifold. Show that for any $(n-1)$-form ω on M there exists a point $p \in M$ such that $d\omega(p) = 0$.

Problem 4: Let $\Psi_t : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the right-handed rotation about the z-axis through t radians. Let $Y = y \cdot \frac{\partial}{\partial z} - z \cdot \frac{\partial}{\partial y}$.

(a) Compute the vector field corresponding to the flow Ψ_t. (Call it X.)

(b) Compute the flow of Y. (Call it Φ_t.)

(c) Compute $\Psi_{-t} \circ \Phi_{-t} \circ \Psi_t \circ \Phi_t$ and the corresponding vector field.

(d) Compute $[X, Y]$.

Problem 5: Let α be a nowhere zero 1-form on $\mathbf{R}^3$.

(a) Given that there are smooth functions, $\lambda : \mathbf{R}^3 \longrightarrow (0, \infty)$ and $f : \mathbf{R}^3 \longrightarrow \mathbf{R}$ such that $\alpha = \lambda\, df$, prove $\alpha \wedge d\alpha = 0$.

(b) Given that $\alpha \wedge d\alpha = 0$, prove that there exist smooth functions, $\lambda : \mathbf{R}^3 \longrightarrow (0, \infty)$ and $f : \mathbf{R}^3 \longrightarrow \mathbf{R}$ such that $\alpha = \lambda\, df$. (*Hint:* Consider $\ker(\alpha) := \{X \in T\mathbf{R}^3 \mid \alpha(X) = 0\}$.)

Problem 6: Give an example of a smooth map between differentiable manifolds such that the inverse image of one point is a differentiable manifold, and the inverse image of a different point is not.

Problem 7: Let ω be an r-form on a differentiable manifold M. Assume that $\int_\Sigma \omega = 0$ for every submanifold Σ of M diffeomorphic to S^r. Show that ω is closed.

Problem 8: Consider the map

$$f : \mathrm{Mat}(2 \times 2, \mathbf{R}) \longrightarrow \mathrm{Mat}(2 \times 2, \mathbf{R}), \qquad X \mapsto X^2 - 3X.$$

Describe the critical points of f.