Differential Topology — Spring 2014

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Problem 1:

Let $M \subset \mathbf{R}^{n-1}$ be an $(n-1)$-dimensional submanifold of $\mathbf{R}^n$. Assume that there exists a differentiable map $\nu : M \longrightarrow \mathbf{R}^n$ such that $\nu(p) \notin T_pM$ for all $p \in M$. Show that M is orientable.

Problem 2: Consider

$$\omega = 3z\, dy \wedge dz + (x^2 + y^2)\, dz \wedge dx + xz\, dx \wedge dy$$

on $\mathbf{R}^3$ and the 2-dimensional submanifold

$$M = \{(x, y, z) \in \mathbf{R}^3 \mid z = xy\}.$$

Determine $\int_A \omega$, where $A = \{(x, y, z) \in M \mid |x| \leq 1,\ |y| \leq 1\}$.

(*Note*: Choose an orientation on M; it is uncritical that the boundary of A is not smooth at the four corners.)

Problem 3:

(a) Let $f_1, \ldots, f_n$ be homogenous polynomials of degree m in the coordinates $x_1, \ldots, x_n$ of $\mathbf{R}^n$ and

$$\omega := \sum_{i=1}^{n} (-1)^{i-1} f_i\, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n.$$

Let $S^{n-1} \subset \mathbf{R}^n$ be the unit sphere. Show that if m is even, then $\int_{S^{n-1}} \omega = 0$. (*Hint:* Consider the antipodal map $x \mapsto -x$ of $\mathbf{R}^n$.)

(b) Prove that the $(n-1)$-dimensional real projective space $\mathbf{RP}^{n-1}$ is orientable if and only if n is even. (*Hint:* Oberserve that the antipodal map on the $(n-1)$-sphere S^{n-1} is orientation preserving if and only if n is even.)

Problem 3: (*Spherical coordinate system*)

Let $\Phi : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be defined by

$$(r,\, \vartheta,\, \varphi) \mapsto (x, y, z) = (r \sin\vartheta \cos\varphi,\, r\sin\vartheta\sin\varphi,\, r\cos\vartheta).$$

It defines a diffeomorphism

$$\Phi : T := \mathbf{R}^*_+ \times (0,\pi) \times (-\pi,\pi) \longrightarrow \Omega := \mathbf{R}^3 \setminus \{(x,y,z) \in \mathbf{R}^3 \mid x < 0,\, y = 0\}.$$

We also denote by $r,\, \vartheta,\, \varphi \to \mathbf{R}$ the components of the inverse of Φ.

a) Show that the function ϑ and the differential form $d\phi$ can be extended from Ω to $\mathbf{R}^3 \setminus (0 \times 0 \times \mathbf{R})$ and the differential form $r^3 \sin\vartheta\, d\vartheta \wedge d\varphi$ can be extended to the whole $\mathbf{R}^3$.

b) Show that for a compact regular domain $K \subset \mathbf{R}^3$ one has

$$\mathrm{Vol}_3(K) = \frac{1}{3} \int_{\partial K} r^3 \sin\vartheta\, d\vartheta \wedge d\varphi.$$

(Here, the volume $\mathrm{Vol}_3(K)$ is defined as $\int_K dx \wedge dy \wedge dz$.)

c) Let $A \subset S^2$ be a compact regular domain inside the 2-sphere which does neither contain the north pole $P_N = (0,0,1)$ not the south pole $P_S = (0,0,-1)$. Show that

$$\mathrm{Vol}_2(A) = 2k\pi - \int_{\partial A} \cos\vartheta\, d\varphi$$

where $k = 0,\, 1,\, 2$, depending on A containing none, one or two of the points $\{P_N,\, P_S\}$, respectivly. (Here, the volume $\mathrm{Vol}_2(A)$ is defined as

$$c \cdot \int_A x dy \wedge dz - y dx \wedge dz + z\, dx \wedge dy$$

with the constant c chosen such that $\mathrm{Vol}_2(S^2) = 4\pi$.)