Differential Topology — Spring 2014

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Problem 1: (*Spherical coordinate system*)

Let $\Phi : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be defined by $(r,\, \vartheta,\, \varphi) \mapsto (x, y, z) = (r \sin\vartheta \cos\varphi,\; r \sin\vartheta \sin\varphi,\; r\cos\vartheta)$. In an open set $V \subset \mathbf{R}^3$ let

$$\begin{aligned}\omega_1 &= f_1\, dx + f_2\, dy + f_3\, dz\\ \omega_2 &= F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy\\ \omega_3 &= \Sigma\, dx \wedge dy \wedge dz.\end{aligned}$$

Let $U = \Phi^{-1}(V)$ and

$$\begin{aligned}\Phi^*\omega_1 &= g_1\, dr + g_2\, d\vartheta + g_3\, d\varphi\\ \Phi^*\omega_2 &= G_1\, d\vartheta \wedge d\varphi + G_2\, d\varphi \wedge dr + G_3\, dr \wedge d\vartheta\\ \Phi^*\omega_3 &= \Xi\, dr \wedge d\vartheta \wedge d\varphi.\end{aligned}$$

Compute the functions $g_i,\, G_i,\, \Xi : U \longrightarrow \mathbf{R}$.

Problem 2:

(a) Consider on $\mathbf{R}^3$ the 2-form

$$\omega = 2xz\, dy \wedge dz + dz \wedge dx - (z^2 + e^x)\, dx \wedge dy.$$

Show that ω is closed and determine a 1-form η with $d\eta = \omega$.

(b) Consider on $\mathbf{R}^2 \setminus \{0\}$ the 1-form

$$\omega = \frac{x\, dy - y\, dx}{x^2 + y^2}$$

Show that ω is closed but not exact.

(*Hint:* Solve $\omega = d\eta$ locally and show that η cannot be extended to a globally defined 1-form.)

Problem 3:

Two differentiable maps f, $g : M \longrightarrow N$ between differentiable manifolds are said to be differentiable homotopic if there exists an open subset $V \subset \mathbf{R} \times M$ containing $[0,1] \times M \subset V$ and a differentiable map $F : V \longrightarrow N$ such that $f(x) = F(0,x)$ and $g(x) = F(1,x)$ for $x \in M$.

Prove that diffentiable homotopic maps f and g induce the same map $f^* = g^* : H^*_{\mathrm{DR}}(N) \longrightarrow H^*_{\mathrm{DR}}(M)$ between the de Rham cohomology groups.

(*Hint:* Construct maps $h_r : \mathcal{E}^r(N) \longrightarrow \mathcal{E}^{r-1}(M)$ for all r such

$$h_{r+1} \circ d + d \circ h_r = f^* - g^*.)$$

Problem 4: (*Symplectic structure on cotangent bundle*)

Let M be a n-dimensional differentiable manifold and $\pi : T^*M \longrightarrow M$ its cotangent bundle. Let $\theta \in \mathcal{E}(T^*M)$ be the 1-form on T^*M which sends a vector in $T(T^*M)$ to $\mathrm{ev}(\sigma, T\pi)$ where $\sigma : T(T^*M) \longrightarrow T^*M$ is the natural projection, $T\pi : T(T^*M) \longrightarrow TM$ is the differential of π and $\mathrm{ev} : T_p^*M \times T_pM \longrightarrow \mathbf{R}$ is the natural evaluation map of a cotangent vector at a tangent vector for a point $p \in M$. Note that this is well-defined since for $u \in T(T^*M)$ the elements $\sigma(u)$ and $T\pi(u)$ are in the cotangent resp. tangent space of the point $\pi(\sigma(u)) \in M$.

Consider the closed 2-form $\omega = d\theta \in \mathcal{E}^2(T^*M)$.

(a) Show that in local coordinates $x_1, \ldots, x_n$ around $p \in M$ one has

$$\omega = x_1 \wedge y_1 + \cdots + x_n \wedge y_n$$

where $y_1 = dx_1, \ldots, y_n = dx_n$ and the x_i, y_i are the used local coordinates for T^*M.

(b) Show that $\omega^n := \underbrace{\omega \wedge \omega \wedge \ldots \wedge \omega}_{n-\text{times}}$ is a nowwhere vanishing $2n$-form on T^*M.