Differential Topology — Spring 2014

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Problem sheet 5   March 6, 2014

Problem 1: Show that for a differentiable manifold M and $p \in M$ the map

$$\mathcal{E}(M) \longrightarrow \mathcal{E}_p(M), \qquad f \mapsto f_p$$

which maps a function to its germ at p is surjective.

Problem 2:

(a) Show that for each n there exists a differentiable map $f : \mathbf{R} \longrightarrow \mathbf{R}^n$ so that for each $k \in \mathbf{N}$ the set $\{f(t) \mid t \geq k\}$ contains all points for which all coordinates are rational.

(b) Find a function $\delta : \mathbf{R} \longrightarrow \mathbf{R}$, $\delta > 0$, and for each n a differentiable map $f : \mathbf{R} \longrightarrow \mathbf{R}^n$, so that for no embedding $g : \mathbf{R} \longrightarrow \mathbf{R}^n$ one has $|g - f| < \delta$

Problem 3: For a compact manifold M^m it is easy to prove an embedding theorem without regard to the dimension. One chooses a finite good atlas $\{\phi_i \mid i = 1, \ldots, r\}$, a bump function ψ for $\overline{B_1(0)}$ with support in $\overline{B_2(0)}$ and one sets $\psi_i = \psi \circ \phi_i : M \longrightarrow \mathbf{R}$ and $k_i = \psi \cdot \phi_i : M \longrightarrow \mathbf{R}^m$.

Show that the map

$$M \longrightarrow \prod_{i=1}^{r} \mathbf{R}^m \times \prod_{i=1}^{r} \mathbf{R}, \qquad p \mapsto (k_1(p), \ldots k_r(p), \psi_1(p), \ldots, \psi_r(p))$$

is an embedding without using any further results from the lecture.

Problem 4: Let X, Y and Z be differentiable vector fields on a differentiable manifold M.

(a) Prove that $[X, Y]$ defined by $[X, Y]_p(f_p) = X_p(Y(f_p)) - Y_p(X(f_p))$ for $f_p \in \mathcal{E}_p(M)$ is indeed a vector fieldon M.

(b) For $f,\, g \in \mathcal{E}(M)$ show that

$$[fX, gY] = fg[X,Y] + fX(g)Y - gY(f)X.$$

(c) Show that $[X,Y] = -[Y,X]$.

(d) Show that $[X,[Y,Z]] = [[X,Y],Z] - [Y,[X,Z]]$.