Differential Topology — Spring 2014

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Problem sheet 3 February 13, 2014

Problem 1: Let X be a set and $\mathcal{A} = \{\phi : U \subset X \longrightarrow V \subset \mathbf{R}^n\}$ be a system of maps with the following properties:

- The $\phi \in \mathcal{A}$ are bijective.
- The images $V = \phi(U)$ are open subsets of $\mathbf{R}^n$ for all $\phi \in \mathcal{A}$
- For each $x \in X$ there exists a $\phi \in \mathcal{A}$, $\phi : U \longrightarrow V$, with $x \in U$.
- Given ϕ_i, $\phi_j \in \mathcal{A}$ the maps $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$ are homeomorphisms.

Show that there is a unique topology on X such that the $\phi \in \mathcal{A}$ are homeomorphisms. (Thus $\mathcal{A}$ makes X into a topological manifold, besides the Hausdorff and 2nd-countability property.)

Problem 2: Prove that the Brieskorn manifold

$$X = \{(z_1, \ldots, z_5) \in \mathbf{C}^5 \mid |z_1|^2 + \cdots + |z_5|^2 = 1,\ z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^5 = 0\}$$

is a 7-dimensional differentiable manifold.

Problem 3: Let M, N and X be differentiable manifolds and $f : M \longrightarrow X$, $g : N \longrightarrow X$ be differentiable maps. Assume that for all points $p \in M$ and $q \in N$ with $f(p) = g(q) = a \in X$ one has

$$T_p f(T_p M) + T_q g(T_q N) = T_a X$$

Show that the fibre product of f and g:

$$\{(p, q) \in M \times N \mid f(p) = g(q)\}$$

is a differentiable manifold.

Problem 4: (from this years qualifying exam):

In this problem all manifolds are assumed to be smooth.

(a) Define what it means for two submanifolds Y and Z of a manifold X to be transversal.

(b) Recall that an affine subspace of $\mathbf{R}^n$ is a translate of a linear subspace, and that affine subspaces are thus trivially seen to be submanifolds. Characterize, with proof, which affine subspaces of $\mathbf{R}^3$ are transversal to the unit sphere $\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$.

(c) Recall that if Y and Z are transversal submanifolds of X, then $Y \cap Z$ is a submanifold of Y with codimension in Y equal to the codimension of Z in X. Show using an example in which X is $\mathbf{R}^3$, Y is the unit sphere, and Z is an affine subspace of $\mathbf{R}^3$ that the conclusion need not hold in the absence of transversality.