Differential Topology — Spring 2014

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Problem 1: Prove the following universal properties for the differentiable sum and product:

(a) There are two differentiable maps $\iota_i : M_i \longrightarrow M_1 \cup M_2$ (the inclusions) such that a map $f : M_1 \cup M_2 \longrightarrow N$ into a manifold N is differentiable if and only if the restrictions $f \circ \iota_i$ are differentiable.

(b) There are two differentiable maps $\pi_i : M_1 \times M_2 \longrightarrow M_i$ (the projections) such that a map $f : N \longrightarrow M_1 \times M_2$ starting from a manifold N is differentiable if and only if the projections $\pi_i \circ f$ are differentiable.

(c) Explain why the properties (a) and (b) characterize $M_1 \cup M_2$ respectively $M_1 \times M_2$ up to diffeomorphism among manifolds.

Problem 2: Let $M(m \times n, \mathbf{R})$ be the vector space of real $m \times n$-matrices with real entries, and $M_r(m \times n, \mathbf{R})$ the subset of matrices of rank r. Prove that $M_r(m \times n, \mathbf{R})$ is a submanifold of $M(m \times n, \mathbf{R})$ of dimension $m \cdot n - (m - r) \cdot (n - r)$ for $r \leq \min\{m, n\}$.

Hint: a typical chart domain around a point of $M_r(m \times n, \mathbf{R})$ is given by the set $U \subset M(m \times n, \mathbf{R})$ of matrices of the form

$$\begin{pmatrix} A & AB \\ D & DB + C \end{pmatrix}, \quad A \in M(r \times r, \mathbf{R}), \quad \det(A) = 0.$$

Such a matrix lies in $M_r(m \times n, \mathbf{R})$ if and only if $C = 0$.

Problem 3: Show that if the map $f : S^n \longrightarrow \mathbf{R}$ is differentiable, then there exist two different points $p,\, q \in S^n$, so that $T_p(f)$ and $T_q(f)$ are both 0.

Problem 4: Let $\mathcal{E}^{(n)} := \mathcal{E}_0(\mathbf{R^n})$ be the algebra of germs of differentiable functions on $\mathbf{R}^n$ at the point $0 \in \mathbf{R^n}$.

(a) Show that $\mathbf{m}_n := \{\phi_0 \in \mathcal{E}^{(n)} \mid \phi_0(o) = 0\}$ is the only maximal ideal of $\mathcal{E}^{(n)}$.

(b) Show that $\mathbf{m}_n$ is generated by the germs of the coordinate functions $x_1, \ldots, x_n$.

(c) Show that the ideal $\mathbf{m}_n^k$ (i.e., the ideal generated by the products $a_1 \cdots a_k$ with $a_i \in \mathbf{m}_n$ for $i = 1, \ldots, k$) is the ideal of germs of functions for which all partial derivatives of order $< k$ vanish at the origin.

(d) Show that for a germ $f_0 \in \mathcal{E}(\mathbf{R}^n, \mathbf{R}^m)$ with $f_0(0) = 0$ the induced map $f^* : \mathcal{E}^{(m)} \longrightarrow \mathcal{E}^{(n)}$ satisfies $f^*(\mathbf{m}_m) \subset \mathbf{m}_n$ and so one obtains a linear map $\bar{f}^* : \mathbf{R}^m \cong \frac{\mathbf{m}_m}{\mathbf{m}_m^2} \longrightarrow \frac{\mathbf{m}_n}{\mathbf{m}_n^2} \cong \mathbf{R}^n$ which is given by the transpose of the Jacobi matrix ${}^t D f_0(0)$.

(e) Show that the dual space $(\frac{\mathbf{m}_n}{\mathbf{m}_n^2})^*$ can be identified with the tangent space $T_0(\mathbf{R}^n)$.