Differential Topology — Spring 2014

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Problem sheet 1 January 22, 2014

Problem 1: Provide the surface of a cube $\{x \in \mathbf{R}^{n+1} \mid \max_{i=0,\ldots,n} |x_i| = 1\}$ with the structure of a differentiable manifold.

Problem 2: (a) Let M be a differentiable manifold and $\tau : M \longrightarrow M$ a fixed point free involution (i.e., τ is a diffeomorphism with $\tau \circ \tau = \mathrm{id}_M$ and $\tau(p) \neq p$ for all $p \in M$). Show that the quotient space M/τ obtained from M by identifying points mapped to each other under τ possesses exactly one differentiable structure such that the projection $\pi : M \longrightarrow M/\tau$ is locally diffeomorphic.

(b) Prove that the real projective space $\mathbf{RP}^n = (\mathbf{R}^{n+1} \setminus \{0\})/\sim$ with $x \sim y$ exactly if there is a $\lambda \in \mathbf{R} \setminus \{0\}$ such that $\lambda x = y$ has the structure of a differentiable manifold.

Problem 3: (a) Show that S^n is a submanifold of $\mathbf{R}^{n+1}$.

(b) Show that if the n-dimensional manifold M is a product of spheres then there exists an embedding $M \longrightarrow \mathbf{R}^{n+1}$.

Problem 4: Let M be a differentiable manifold and $\mathcal{E}(M)$ be the set of differentiable functions. Show:

(a) $\mathcal{E}(M)$ is an algebra (i.e., a vector space with a bilinear associative product) under the natural addition and multiplication of functions.

(b) A differentiable map $f : M \longrightarrow N$ induces an algebra homomorphism $f^* : \mathcal{E}(N) \longrightarrow \mathcal{E}(M)$, $h \mapsto h \circ f$ with the properties $\mathrm{Id}_M^* = \mathrm{Id}$ and $(g \circ f)^* = f^* \circ g^*$.

(c) For a point $p \in M$ the set $\mathcal{M}_p = \{h \in \mathcal{E}(M) \mid h(p) = 0\}$ is a maximal ideal of $\mathcal{E}(M)$.

(d) If M is compact and $\mathcal{M}$ is a maximal ideal of $\mathcal{E}(M)$, then there exists a $p \in M$ such that $\mathcal{M} = \mathcal{M}_p$.