Higher Algebra 2 — Spring 2006

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Problem 1: Let K/k be a Galois extension with Galois group G. Let H be a subgroup of G and denote the normalizer of H in G by N.

(a) Prove that N acts on K^H by automorphisms and that the field extension K^H/K^N is Galois with Galois group N/H.

(b) Prove that if $\varphi \in \mathrm{Aut}(K^H/k)$ then φ fixes K^N, i.e., K^N is the smallest subfield L of K^H containing k such that K^H/L is Galois.

Problem 2: Let K_1 and K_2 be two extension fields of a field k contained both in a field K, and assume that the two field extensions K_1/k and K_2/k are both Galois.

(a) Prove that $K_1 \cap K_2/k$ is a Galois extension.

(b) Prove that K_1K_2/k is a Galois extension with Galois group

$$\{(\sigma, \tau) \mid \sigma|_{K_1\cap K_2} = \tau|_{K_1\cap K_2}\} < \mathrm{Aut}(K_1/k) \times \mathrm{Aut}(K_2/k).$$

Problem 3: Let p be a prime and n be a natural number.

(a) Prove that there exists an irreducible polynomial of degree n over $\mathbf{F}_p$.

(b) Prove that the product of all monic irreducible polynomials in $\mathbf{F}_p[x]$ of degree d where d runs through all the divisors of n equals $x^{p^n} - x$.

Problem 4:

(a) Find a primitive element for the field extension $\mathbf{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})/\mathbf{Q}$.

(b) Let $L = \mathbf{F}_3(x, y)$ be the field of rational functions in the variables x and y over the finite field $\mathbf{F}_3$ with three elements. Let $K = \mathbf{F}_3(x^3, y^3) \subset L$. Prove that the field extension L/K is not simple. (Hint: Show that $[L : K] = 9$, but any simple extension of K by an element of L has degree at most 3.)