Higher Algebra 2 — Spring 2006

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Problem 1: (a) Let K be a field of characteristic $p > 0$. Let L be a finite extension of K and suppose that $[L : K]$ is prime to p. Show that L is separable over K.

(b) Let K/k be a separable extension, let L/k be an arbitrary extension and assume K and L are subfields of a common extension field. Prove that the field extension KL/L is separable.

Problem 2: Let K/k be an algebraic field extension, where k is of characteristic $p > 0$. An element $a \in K$ is called *purely inseparable* over k, if its minimal polynomial has only one different root. Prove that the following statements are equivalent:

1. The extension K/k is purely inseparable, i.e., every element $a \in K$ is purely inseparable over k.
2. The extension K/k has separable degree $[K : k]_s = 1$.
3. For every element $a \in K$ there exists an $n \in \mathbf{N} \cup \{0\}$ such that $a^{p^n} \in k$.
4. The minimal polynomial of every element $a \in K$ is of the form $x^{p^n} - b$, where $b \in k$ and $n \in \mathbf{N} \cup \{0\}$.
5. There exists a set of generators of K over k which are purely inseperable over k.

Problem 3: Determine the splitting field of $f(x) = x^4 - 4\,x^2 - 1 \in \mathbf{Q}[x]$ in $\mathbf{C}$, determine its Galois group over $\mathbf{Q}$, and describe the lattices of its subfields and of subgroups of the Galois group.

Problem 4: Show that the Galois group of $f(x) = x^5 - 4\,x + 2 \in \mathbf{Q}[x]$ over $\mathbf{Q}$ is the symmetric group S_5.