Higher Algebra 2 — Spring 2006

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Problem 1: Let k be a field of characteristic different from 2.

(a) Show that K/k is an extension of degree 2 if and only if K of the form $k(\sqrt{D})$ where D is an element of k which is not a square in k. Here, $\sqrt{D}$ denotes a root of the polynomial $x^2 - D$.

(b) Let D_1 and D_2 be elements of k, neither of which is a square in k. Prove that $k(\sqrt{D_1}, \sqrt{D_2})$ is of degree 4 over k if $D_1 D_2$ is not a square in k and is of degree 2 otherwise.

Problem 3: Determine the splitting fields and its degrees over $\mathbf{Q}$ of the following polynomials:

(a) $x^4 - 2$,
(b) $x^4 + 2$,
(c) $x^4 + x^2 + 1$,
(d) $x^6 - 4$.

Problem 3: Let K_1 and K_2 be two subfields of a field K. The composite field K_1K_2 of K_1 and K_2 is the intersection of all subfields of K containing both K_1 and K_2. Let K_1 and K_2 be two extension fields of a field k contained in a field K.

(a) Prove that $[K_1K_2 : k] \leq [K_1 : k][K_2 : k]$. (In particular, K_1K_2/k is finite if K_1/k and K_2/k are both finite.)

(b) Prove that K_1K_2/k is algebraic if K_1/k and K_2/k are both algebraic.

Problem 4: Let K_1 and K_2 be two extension fields of a field k contained in a field K, and assume both are the splitting fields over k of certain polynomials in $k[x]$.

Prove that $K_1 \cap K_2$ is a splitting field over k. (*Hint:* Prove first that K_1K_2 is a splitting field.)