Higher Algebra 2 — Spring 2006

Gerald Hoehn

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Problem 1: Let K be an algebraic closed field of prime characteristic p and let V be an K-vector space of dimension precisely p. Suppose A and B are endomorphisms of V such that $AB - BA = B$. If B is invertible, prove that V has a basis $\{v_1, v_2, \ldots, v_p\}$ of eigenvectors of A such $Bv_i = v_{i+1}$ for $1 \leq i \leq p-1$ and $Bv_p = \lambda v_1$ for some nonzero $\lambda \in K$.

Problem 2: Let $\mathrm{GL}_5(\mathbf{Q})$ be the group of invertible 5×5 matrices with rational entries and matrix multiplication as product. Find representatives for all conjugacy classes of elements of order 10 in $\mathrm{GL}_5(\mathbf{Q})$.

Problem 3: Find a canonical form for $n \times n$ matrices with entries in the field of real numbers corresponding to the elementar divisor normal form of modules over principal domains, i.e., similar to the Jordan canonical form for algebraic closed fields. One can use that the irreducible real polynomials are either linear or quadratic (with complex conjugated complex roots).

Problem 4: (a) Let $f(z) = \sum_{n=0}^{\infty} a_n \cdot z^n$ be a power series with complex coefficients and radius of convergence $r > 0$. Let A be a $k \times k$-matrix with real or complex entries. Prove that the series $\sum_{n=0}^{\infty} a_n \cdot A^n$ is convergent in $\mathrm{Mat}(k \times k, \mathbf{C}) \cong \mathbf{R}^{2k^2}$ (with respect to any of the equivalent nontrivial norms) if the largest absolute value of an eigenvalue of A is less than r.

(b) Let A, $B \in \mathrm{Mat}(k \times k, \mathbf{C})$ be two commuting matrices (i.e, $AB = BA$). Prove that $\exp(A + B) = \exp(A) \cdot \exp(B)$, where $\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot z^n$ is the exponential function.

(c) Prove for $A \in \mathrm{Mat}(k \times k, \mathbf{C})$ that $\det(\exp(A)) = e^{\mathrm{tr}(A)}$, where $\mathrm{tr}(A)$ is the trace of the matrix A (the sum of the diagonal entries of A).

(d) Find the derivative of $\exp : \mathrm{Mat}(k \times k, \mathbf{C}) \longrightarrow \mathrm{Mat}(k \times k, \mathbf{C})$.

(e) Show that in a neighbourhood of the identity matrix an inverse function log of exp exists (i.e, $\log(\exp(A)) = A$) and that $\log(A) = \sum_{n=0}^{\infty} b_n \cdot A^n$ with a convergent power series $g(z) = \sum_{n=0}^{\infty} b_n \cdot z^n$.

Problem 5*: Expand the differentiable map

$$\mathrm{inv} : \mathrm{GL}_k(\mathbf{R}) \subset \mathrm{Mat}(k \times k, \mathbf{R}) \longrightarrow \mathrm{Mat}(k \times k, \mathbf{R}), \quad A \mapsto A^{-1}$$

into a Taylor series around the identity matrix.