# Higher Algebra 2 — Spring 2006

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**Problem 1:** Let $x_1, \ldots, x_n$ be elements of a field $K$. Prove the following formula for the *Vandermonde determinant:*

$$\begin{vmatrix} 1 & x_1 & \ldots & x_1^{n-1} \\ 1 & x_2 & \ldots & x_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \ldots & x_n^{n-1} \end{vmatrix} = \prod_{i<j}(x_j - x_i).$$

**Problem 2:** Prove the following theorem about the rank of a matrix: Let $A \in \mathrm{Mat}(n \times m, K)$ and let $r$ be the largest non-negative integer such that there exist an $r \times r$ submatrix $A'$ of $A$ with $\det A' \neq 0$. Then $\operatorname{rank} A = r$. Here, a submatrix of $A$ is a matrix obtained by removing an abitrary number of rows and columns from $A$.

**Problem 3:** Suppose $A$ is an $n \times n$ matrix with real entries such that the diagonal elements are all positive, the off-diagonal elements are all negative and the row sums are all positive. Prove that $\det A \neq 0$.

**Problem 4:** Compute the determinant of the matrix

$$A = \left(\begin{array}{c|c} B_1 & C \\ \hline 0 & B_2 \end{array}\right)$$

in terms of the three submatrices $B_1$, $B_2$ and $C$.

**Problem 5*:** Let $A$ be an $n \times n$ matrix. Let $I = \{i_1, \ldots, i_p\}$ be a subset of $\{1, \ldots, n\}$ and let $J = \{j_1, \ldots, j_q\}$ be the complement of $I$ in $\{1, \ldots, n\}$. Denote by $\Gamma$ the set of all permutations $\sigma \in S_n$ such that the restrictions of the map $\sigma$ to $I$ and $J$ are monotone increasing functions. For $\gamma \in \Gamma$ let $A_\gamma$ be the submatrix of $A$ obtained by removing the rows $j_1, \ldots, j_q$ and columns $\gamma(j_1), \ldots, \gamma(j_q)$ from $A$ and let $A_\gamma^*$ be the submatrix of $A$ obtained by removing the rows $i_1, \ldots, i_p$ and columns $\gamma(i_1), \ldots, \gamma(i_p)$ from $A$. Prove the following *expansion theorem of Laplace:*

$$\det A = \sum_{\gamma \in \Gamma} \epsilon(\gamma) \cdot \det A_\gamma \cdot \det A_\gamma^*.$$