Higher Algebra I — Fall 2005

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Problem 1: Prove the following remark from the lecture about the centralizer and normalizer of a subset S of a group G:

(a) $\mathrm{Cent}(S) < \mathrm{Nor}(S) < G$.

(b) $H < G$ implies $H \triangleleft \mathrm{Nor}(H)$.

(c) $K < G$ and $H \triangleleft K$ implies $K < \mathrm{Nor}(H)$, i.e., $\mathrm{Nor}(H)$ is the largest subgroup of G in which H is normal.

Problem 2: Let $c : G \longrightarrow \mathrm{Perm}(G)$ be the map defined for $a \in G$ by $c(a) : G \to G$, $g \mapsto aga^{-1}$.

(a) Show that c is a homomorphism from G into $\mathrm{Aut}(G) < \mathrm{Perm}(G)$, the subgroup of automorphisms of G.

(b) Is the image of c normal in $\mathrm{Aut}(G)$?

(c) Determine the kernel of c.

Problem 3: (a) Let G be group and H be a group of finite index. Show that there exists a normal subgroup N of G contained in H and also of finite index. (Hint: If $[G : H] = n$, find a homomorphism of G into S_n (the group of permutations of the set $\{1, \ldots, n\}$) whose kernel is contained in H.)

(b) Let G be a group and let H_1, H_2 be subgroups of finite index. Prove that $H_1 \cap H_2$ has finite index.

Problem 4: Determine a composition series of the group S_4, the group of permutations of the set $\{1, 2, 3, 4\}$.