Higher Algebra I — Fall 2005

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Problem 1: Prove Lemma 2 from Chapter 1: In a commutative semigroup H one has $\prod_{i=1}^{n} a_i = \prod_{i=1}^{n} a_{\pi(i)}$ for any $a_1, \ldots, a_n \in H$ and bijective map $\pi : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$.

Problem 2: A *left identity* in a semigroup G is an element e such that $ea = a$ for all $a \in G$. A *left inverse* of $a \in G$ is an element $x \in G$ such that $xa = e$, with e a left identity. Prove that a semigroup which has a left identity and every element has a left inverse is in fact a group and the left identity is an identity and a left inverse is an inverse.

Problem 3: (i) Is the additive group of integers isomorphic to the additive group of rationals?

(ii) Is the additive group of rationals isomorphic to the multiplicative group of non-zero rationals?

Problem 4: Prove part (iv) and (v) of the list of subgroup examples:

(iv) H is a subgroup of the additive group of integers if and only if there exists a $n \in \mathbf{Z}$ such that $H = n\mathbf{Z} := \{nk \mid k \in \mathbf{Z}\}$.

(v) The set $Q = \{\pm E, \pm I, \pm J, \pm K\}$ with $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, is a subgroup of $\mathrm{GL}_2(\mathbf{C})$ of size 8.

Furthermore, determine all subgroups of the quaternion group Q.

Problem 5*: Generalize Theorem 3 of Chapter 1 to arbitrary (i.e., not necessary abelian) semigroups and groups.