Exam 2 – Introduction to Modern Algebra – Math 512

Monday, November 20, 2006

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1. Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 2 & 3 & 4 & 1 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 2 & 1 & 7 & 3 & 7 & 5 \end{pmatrix}$$

be two permutations in the symmetric group of degree seven.

(a) Find $\sigma\tau^{-1}$.

(b) Write σ as a product of distinct cycles.

(c) What is the order of σ? Explain.

(d) Write σ as a product of transpositions. Is σ an even or an odd permutation?

(e) Determine the number of cosets of $H = \langle\sigma\rangle$ in S_7.

2. (a) Give the multiplication table for the ring $\mathbb{Z}_6$ of residue classes mod 6.

$\cdot$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{0}$						
$\overline{1}$						
$\overline{2}$						
$\overline{3}$						
$\overline{4}$						
$\overline{5}$						

(b) Which elements in $\mathbb{Z}_6$ are invertible with respect to the multiplication? Which elements are zero-divisors?

(c) Determine all ideals in $\mathbb{Z}_6$.

3. Prove that the principal ideal $(\overline{3})$ in $\mathbb{Z}_{54}$ is a maximal ideal.

4. Consider the two ideals $I = (100) = 100\mathbb{Z}$ and $J = (30) = 30\mathbb{Z}$ in the ring of integers. Determine the ideals

(a) I+J,

(b) $I \cap J$,

(c) IJ.

5. Consider the subring $R = \left\{ \left(\begin{smallmatrix} a & c \\ 0 & 5b \end{smallmatrix}\right) \mid a, b, c \in \mathbb{Z} \right\}$ of the ring of 2×2-matrices with integer entries. Answer the following questions and explain:

   (a) Is R a commutative ring?

   (b) Does R have a unit?

   (c) Does R have zero-divisors?

6. Find the greatest common divisor $d(x)$ of the polynomials $f(x) = 9\,x^4 + 12\,x^3 - 5\,x^2 - 12\,x - 4$ and $g(x) = 3x^3 + 5\,x^2 - 4\,x - 4$ in $\mathbb{Q}[x]$. Also, find $a(x), b(x) \in \mathbb{Q}[x]$ with $d(x) = a(x)f(x) + b(x)g(x)$.

7. Determine if the following polynomials are irreducible in $\mathbb{Q}[x]$.

   (a) $f(x) = x^{11} - 1001\,x^5 + 444444\,x^2 - 123123\,x + 231$.

   (b) $g(x) = x^3 + x^2 + x + 2$.

   (c) $h(x) = 1 + 8\,x - 12\,x^3 + 2\,x^4$.

   (d) $k(x) = \frac{2}{3}\,x^4 + 2\,x^2 + \frac{4}{3}$.

8. For a ring R with unit 1, the smallest positive integer n such that $n\,1 = 0$ is called the *characteristic* of R. If no such positive integer n exists, R is said to have characteristic 0.

   (a) Prove that an integral domain R cannot have characteristic 25.

   (b) Let R be a ring with unit which is Boolean, i.e., one has $x^2 = x$ for all $x \in R$. Show that the characteristic of R is 2.