# Exam 1 – Introduction to Modern Algebra – Math 512
## Friday, October 6, 2006

---

1. (a) Compute the g.c.d. $d$ of 105 and 144 by using the Euclidean algorithm.

   (b) Find integers $x$ and $y$ with $105\,x + 144\,y = d$.

2. Suppose that $a \sim b$ if $a + b$ is divisible by 4. Is $\sim$ an equivalence relation on the integers $\mathbf{Z}$? Why?

3. Suppose that $G = \{1, i, j, k, -1, -i, -j, -k\}$ is a group with identity element 1. (Here, $-x$ is just a name of an element different from $x$ and doesn't indicate that $-x$ is the inverse of $x$.)

   (a) Complete the group table

| $\cdot$ | $1$ | $i$ | $j$ | $k$ | $-1$ | $-i$ | $-j$ | $-k$ |
|---|---|---|---|---|---|---|---|---|
| $1$ | $1$ | $i$ | $j$ | $k$ | $-1$ | $-i$ | | $-k$ |
| $i$ | $i$ | $-1$ | | $-j$ | $-i$ | $1$ | | $j$ |
| $j$ | $j$ | $-k$ | | $i$ | | $k$ | | $-i$ |
| $k$ | $k$ | $j$ | $-i$ | $-1$ | $-k$ | | | $1$ |
| $-1$ | $-1$ | $-i$ | $-j$ | $-k$ | $1$ | $i$ | $j$ | $k$ |
| $-i$ | $-i$ | $1$ | $-k$ | | | $-1$ | | |
| $-j$ | $-j$ | $k$ | $1$ | $-i$ | $j$ | $-k$ | | |
| $-k$ | $-k$ | $-j$ | $i$ | $1$ | $k$ | $j$ | | |

.

   (b) Is $G$ abelian?

   (c) Find the inverse of each element of $G$.

   (d) Give the order of each element of $G$.

4. Let $\varphi : G \longrightarrow H$ be a map between two groups $G$ and $H$. List all properties which have to be satisfied in order for $\varphi$ to be an isomorphism.

5. Proof that if $\varphi : G \longrightarrow H$ and $\psi : H \longrightarrow I$ are group homomorphisms, also the composite map $\psi \circ \varphi : G \longrightarrow I$ is a group homomorphism.

6. Let $(\mathbf{Z}_{12}, +)$ be the additive rest class group mod 12.

   (a) List all proper subgroups of $\mathbf{Z}_{12}$.

   (b) Does there exist a group homomorphism from $(\mathbf{Z}_{12}, +)$ to $(\mathbf{Z}_5, +)$ which is onto (i.e, surjective)? You have to justify your answer!

7. Let $M = \left\{ \left( \begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix} \right) \mid a, b, c \in \mathbf{R} \text{ and } ac \neq 0 \right\}$ be the set of invertible upper-triangular $2 \times 2$-matrices with real entries.

   (a) Show that $M$ together with the matrix multiplication as product is a group.

   (a) Is the determinant map $\det : M \longrightarrow \mathbf{R}^*$, $\left( \begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix} \right) \mapsto \det(\left( \begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix} \right)) = ac$, a group homomorphism?

8. Let $G$ be a group

   (a) Show that $N = \{g_1 h_1 g_1^{-1} h_1^{-1} ... g_n h_n g_n^{-1} h_n^{-1} \mid n \in \mathbf{N}, g_1, ..., g_n, h_1, ..., h_n \in G\}$ is a subgroup of $G$.

   (b) Prove that $N \lhd G$, i.e., $N$ is a *normal* subgroup of $G$.