Calculus I - Lecture 9
Applications and Higher Derivatives

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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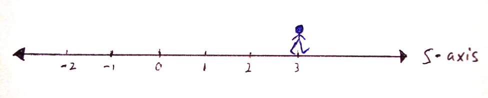
February 19, 2014

Section 3.4 — Rates of Change

Motion along a straight line:



s = position of object on the line = $s(t)$: function of time
t = time

Average velocity over $[t_1, t_2]$

$$v_{\text{ave}} = \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{\Delta s}{\Delta t} = \frac{\text{change in position}}{\text{change in time}}$$

Instantaneous velocity at t_0

$$v_{\text{inst}}(t_0) = \lim_{t \to t_0} v_{\text{ave}} = \lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0} = s'(t_0) = \left. \frac{\mathrm{d}s}{\mathrm{d}t} \right|_{t=t_0}$$

Let $v(t)$ be the (instantaneous) velocity at time t:

$$v(t) = s'(t) = \tfrac{\mathrm{d}s}{\mathrm{d}t}$$

$v(t) > 0$ means object is moving to the right.

$v(t) < 0$ means object is moving to the left.

$v(t) = 0$ means object has stopped.

Example: The position of a car is given by

$$s(t) = 2\,t^2 - 16\,t + 35 \quad (s \text{ in meters, } t \text{ time in sec.})$$

a) Find $v(t)$.

Solution:
$v(t) = \frac{\mathrm{d}}{\mathrm{d}t}(2\,t^2 - 16\,t + 35) = 2 \cdot 2t - 16 \cdot 1 + 0 = 4\,t - 16$

b) Where does the car start?

Solution: At start $t = 0$.
Position $s(0) = 2 \cdot 0^2 - 16 \cdot 0 + 35 = 35 \text{ m}$.

c) When does the car come to a stop? Where is it at this time?

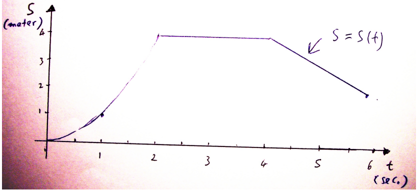
Solution: At stop $v = 0$.
$v(t) = 0 \;\Rightarrow\; 4t - 16 = 0 \;\Rightarrow\; 4t = 16 \;\Rightarrow\; t = 4 \text{ sec}$.
$s(4) = 2 \cdot 4^2 - 16 \cdot 4 + 35 = 32 - 64 + 35 = 3 \text{ m}$.

d) What does the the car after it stops?

Solution: If $t > 4$, $v(t) = 4t - 16 > 0$. It moves to the right.

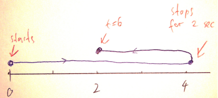
e) Describe the motion on the number line.

Example: The position of an object moving along a straight line is given by the graph below



a) Show the motion of the object along s-axis:

Solution:



b) Find velocity of the object when $t = 1, 3, 5$ sec.

Solution: $v(t) = \frac{\mathrm{d}}{\mathrm{d}t} t^2 = 2t$ for $0 \leq t \leq 2$.

$v(1) = 2 \cdot 1 = 2$ m/sec.

$v(3) = 0$ m/sec, slope $= 0$.

$v(5) = -1$ m/sec, slope $= -1$.

Concept of the Derivative: Suppose that y is a quantity that depends on x, according to the law $y = f(x)$. Then

$f'(x) =$ rate of change of y with respect to x.

Example: Population growth. Let $P = P(t)$ denote the size of a rabbit population as a function of time (days).

a) What measures $P'(t)$

Solution: $P'(t) =$ Rate of change of population with respect to time (in rabbits per day).

b) Interpret $P'(100) = -5$.

Solution: At the 100th day, the rabbit population is decreasing at a rate of 5 rabbits per day.

c) What would it mean to say that $P'(t) = 0$ for $10 \leq t \leq 20$?

Solution: Population is constant between 10th and 20th day.

The **units** of a rate of change $R = \frac{\mathrm{d}y}{\mathrm{d}x}$ are the units of the dependent variable y divided by the units of the independent variable x.

Example: Let y be the amount of snowfall in Kansas in inches per year and x be the average temperature in the winter in degree Celsius below zero. What are the units of $\frac{\mathrm{d}y}{\mathrm{d}x}$?

Solution: : inches/(year$\cdot^{\circ}$C)

Example: Find the rate of change of the area of a circle with respect to the radius.

Solution:

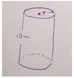
$$A = \pi r^2$$

$$\frac{\mathrm{d}A}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}(\pi r^2) = \pi \cdot 2r = 2\pi r \qquad \text{(Circumference)}$$

Example: Find the rate of change of the volume of a cylinder with respect to r if $h = 10m$.

Solution:

$$V = \pi r^2 h = \pi d^2 \cdot 10 = 10\pi r^2$$



$$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}(10\pi r^2) = 10\pi \cdot 2r = 20\pi r$$

Example: Find the rate of change of the volume of a sphere with respect to its radius r.

Solution:

$$V = \frac{4}{3}\pi r^3$$

$$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi \cdot 3r^2 = 4\pi r^2 \qquad \text{(surface area of sphere)}$$

Section 3.5 – Higher Derivatives

Let $y = f(x)$.

$f'(x) = y' = \dfrac{\mathrm{d}y}{\mathrm{d}x} =$ First derivative of $f(x)$

$f''(x) = y'' = \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} := \dfrac{\mathrm{d}}{\mathrm{d}x}\left(f'(x)\right) =$ Second derivative of $f(x)$

$f'''(x) = y''' = \dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} := \dfrac{\mathrm{d}}{\mathrm{d}x}\left(f''(x)\right) =$ Third derivative of $f(x)$

$\ldots$

Example: Let $f(x) = 2x^5 - 3x^2$. Find $f'(x)$, $f''(x)$ and $f'''(x)$.

Solution:

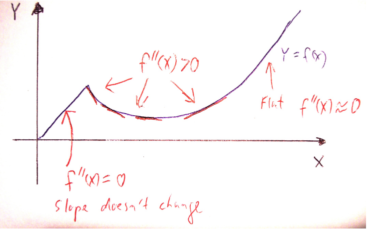
a) $f'(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(2x^5 - 3x^2\right) = 2 \cdot 5x^4 - 3 \cdot 2x = 10x^4 - 6x$

b) $f''(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(10x^4 - 6x\right) = 10 \cdot 4x^3 - 6 \cdot 1 = 40x^3 - 6$

c) $f'''(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(40x^3 - 6\right) = 40 \cdot 3x^2 - 0 = 120x^2$

What is the interpretation of $f''(x)$?

$f''(x)$ measures the rate at which $f'(x)$ is changing, that is, the rate at which the slope of the curve $y = f(x)$ is changing.



$f''(x)$ measures the curvature of the graph!

Motion along a straight line (distance in meters, time in seconds):

$s = s(t)$ is position (m)

$v = s'(t)$ is velocity (m/s)

$a = v'(t) = s''(t)$ is acceleration (m/sec^2)

Example: An object falling from a roof has after t seconds a height h measured in feet given by

$$h = 100 - 16\, t^2.$$

a) Find velocity and acceleration

b) How fast is the object traveling when it hits the ground?

Solution:

a) $v = \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\big(100 - 16\, t^2\big) = -32t(\mathrm{ft}/\mathrm{sec})$

$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\big(-32t\big) = -32\ (\mathrm{ft}/\mathrm{sec}^2)$

b) It hits the ground when $h = 0$: $100 - 16t^2 = 0$or

$t = \sqrt{\frac{100}{16}} = 5/2\ \mathrm{sec}.$

$v(\frac{5}{2}) = -32 \cdot \frac{5}{2} = -80\ \mathrm{ft}/\mathrm{sec}$

It is traveling with a velocity of 80 ft/sec downward.