Calculus I - Lecture 7 - The Derivative

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

Gerald Hoehn (based on notes by T. Cochran)

February 12, 2014

Section 3.1 – Definition of the Derivative

In this section we will give both a **geometric** and an **algebraic** definition of the derivative

Geometric View of the Derivative

Recall, the slope of a line is

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x}$$

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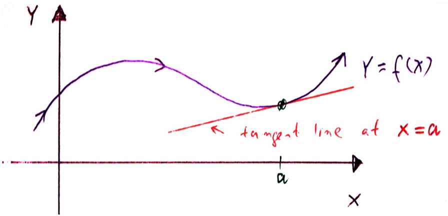

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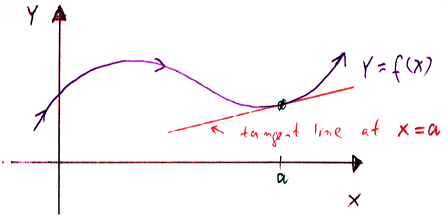

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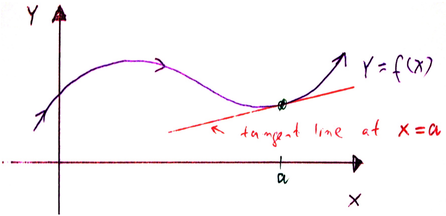

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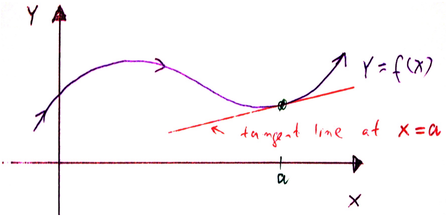

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If the curve is a line segment, the tangent line coincides with the segment.

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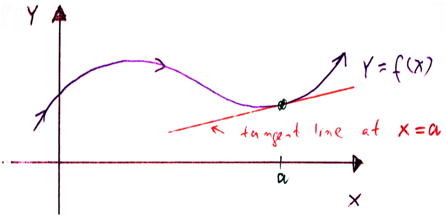

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If the curve is a line segment, the tangent line coincides with the segment.

Slope of a curve at $x = a$ equals m_{tan} = slope of tangent line.

Definition (Derivative — geometric)

The **derivative** *of a function $f(x)$ at $x = a$, denoted $f'(a)$ (pronounced "f prime of a"), is the slope of the curve $y = f(x)$ at $x = a$.*

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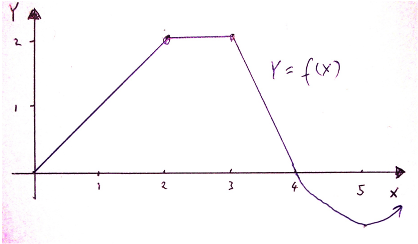
The **derivative** *of a function $f(x)$ at $x = a$, denoted $f'(a)$ (pronounced "f prime of a"), is the slope of the curve $y = f(x)$ at $x = a$.*

$$\begin{aligned} f'(a) &= \text{the derivative of f(x) at } a \\ &= m_{\tan}, \text{ the slope of the tangent line.} \end{aligned}$$

Example: Determine by inspection the following derivatives.



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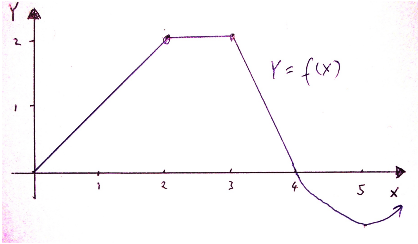

a) $f'(1) =$

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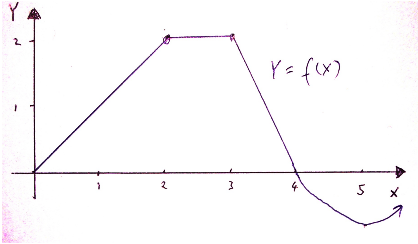

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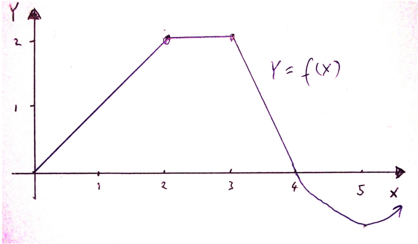

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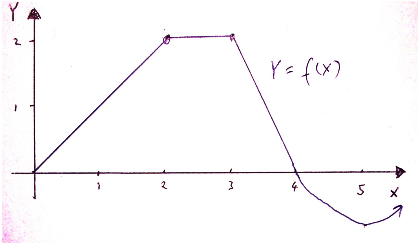

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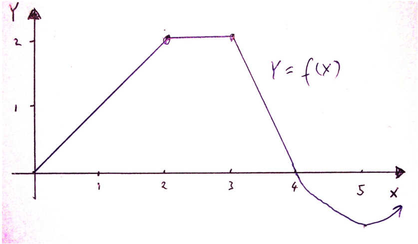

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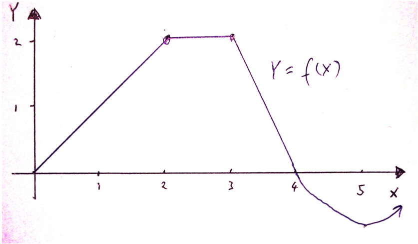

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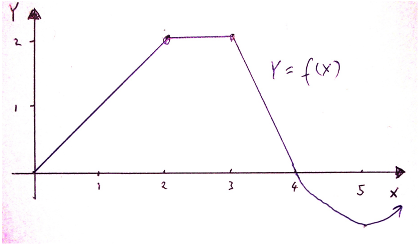

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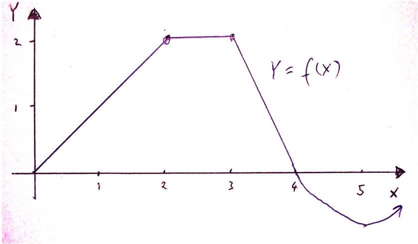

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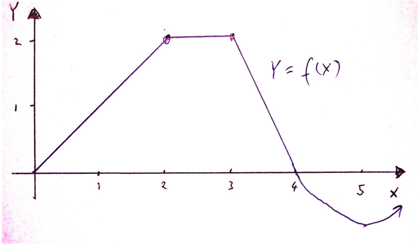

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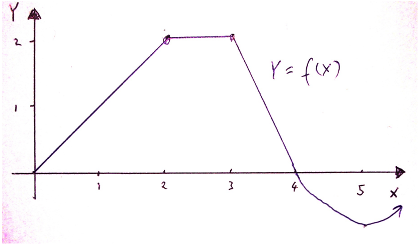

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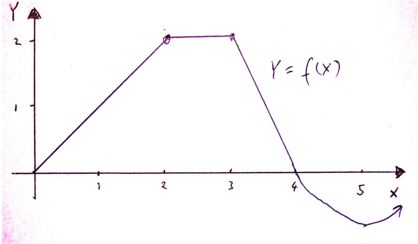

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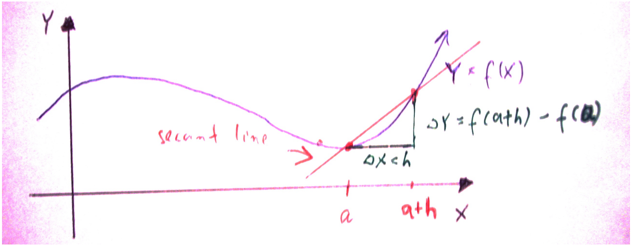
At sharp corners, $f'(a)$ does not exist.

Algebraic View of the Derivative



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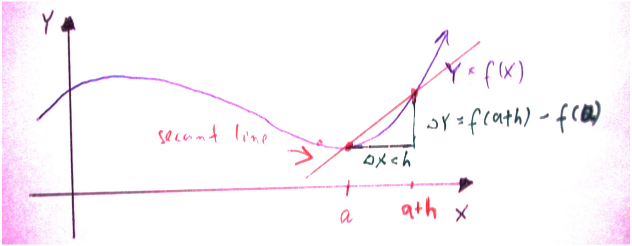

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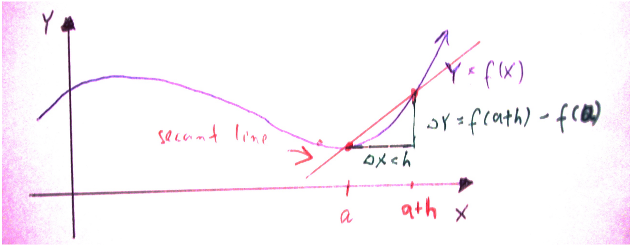

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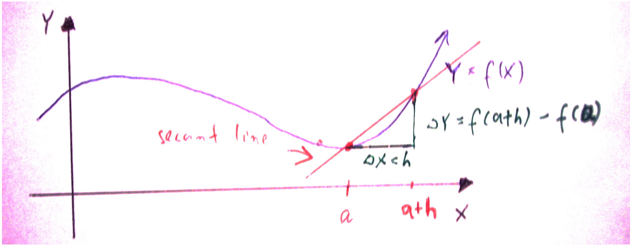

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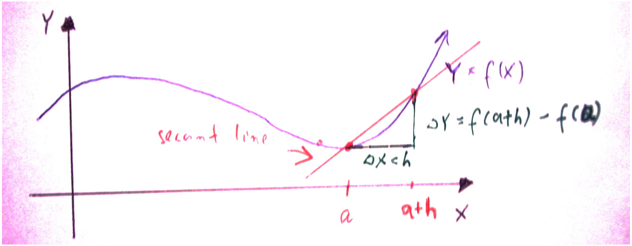

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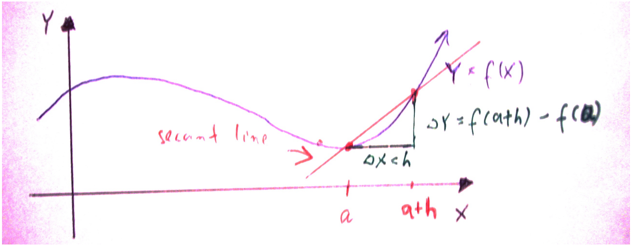

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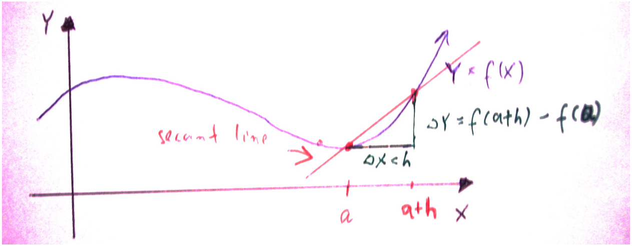

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Memorize this!

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a) Find $f'(2)$ where $f(x) = x^2 - 2x$.

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Solution: a)

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$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

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&= \lim_{h \to 0} \frac{2h + h^2}{h}
\end{aligned}
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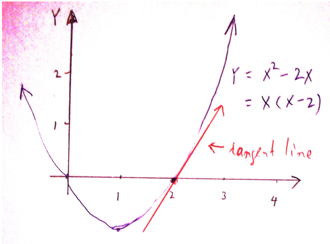
Solution: b)



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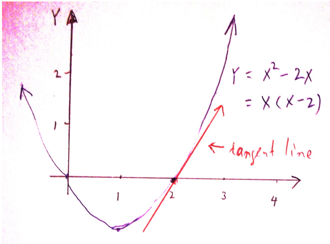

We use point-slope form of the tangent line:

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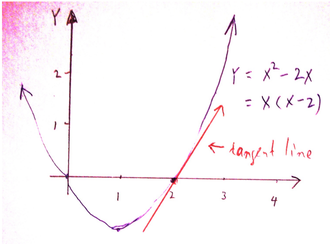

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$$y - y_1 = m(x - x_1)$$

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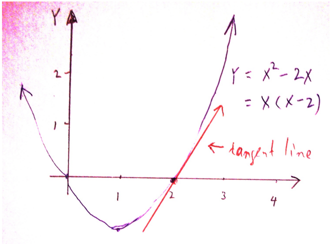

We use point-slope form of the tangent line:

$$y - y_1 = m(x - x_1)$$

$$y - 0 = 2(x - 2) \quad \text{(since } m = 2 \text{ by a))}$$

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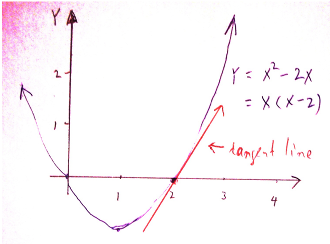

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$$y = 2x - 4$$

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Example: Let $f(x) = \sqrt{x}$. Find $f'(a)$, where a is any value > 0.

Solution:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{a+h} - \sqrt{a})}{h} \cdot \frac{(\sqrt{a+h} + \sqrt{a})}{(\sqrt{a+h} + \sqrt{a})}$$

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&= \lim_{h \to 0} \frac{(a+h) - \sqrt{a}\sqrt{a+h} + \sqrt{a}\sqrt{a+h} - a}{h\left(\sqrt{a+h} + \sqrt{a}\right)}
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$$= \lim_{h\to 0} \frac{h}{h(\sqrt{a+h} + \sqrt{a})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{a+h} + \sqrt{a}} =$$

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We will see short cuts next time.

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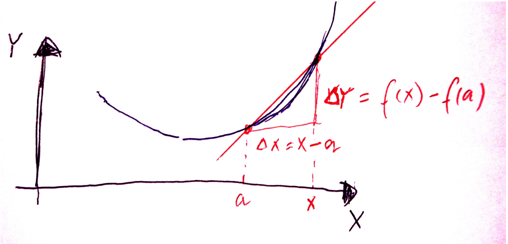

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}.$$

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Note:

We have seen that

1. if $f(x) = \sqrt{x} = x^{1/2}$ then $f'(a) = \frac{1}{2}a^{-1/2}$,
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Theorem (Power rule)

Let n be any real number. If $f(x) = x^n$, then $f'(a) = n \cdot a^{n-1}$ for any real number a where $f(x)$ is defined.

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Since a is arbitrary, we simply replace a with x (a variable) and say

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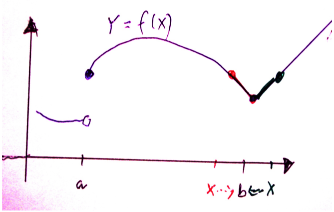

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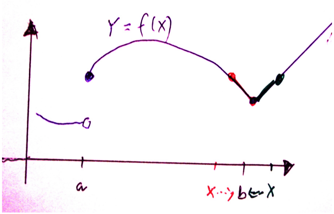

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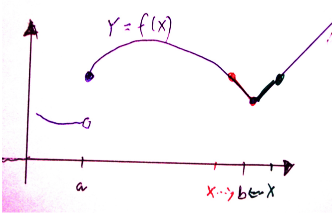

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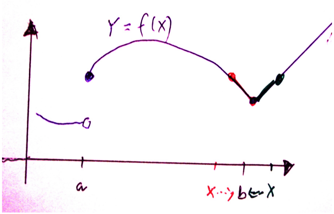

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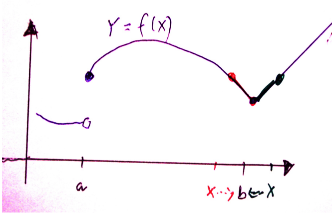

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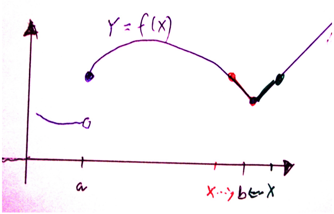

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Thus the two-sided limit $f'(b) = \lim_{x\to b} \frac{f(x)-f(b)}{x-b}$ does not exist.