Calculus I - Lecture 6
Limits D & Intermediate Value Theorem

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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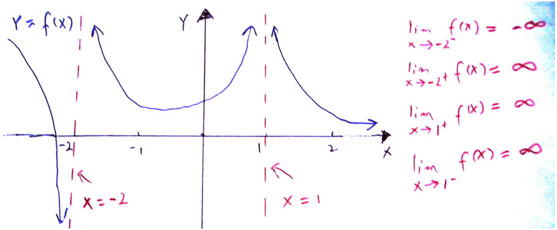
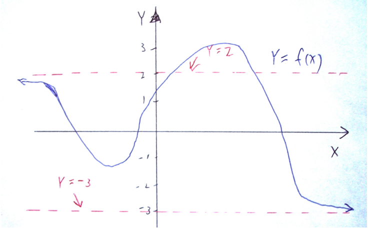
Section 2.7 – Limits at Infinity

Recall: 1) A **vertical asymptote** is a guideline that the graph of $f(x)$ approaches at points where $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$.



2) A **horizontal asymptote** is a guideline that the graph of $f(x)$ approaches at points where $x \to \pm\infty$.



Limits at Infinity

Definition

The limits of a function $f(x)$ **at infinity** are the values $f(x)$ approaches as $x \to \infty$, written $\lim_{x\to\infty} f(x)$, or $x \to -\infty$, written $\lim_{x\to-\infty} f(x)$. They may or may not exist.

Example: In the previous example we find:

$$\lim_{x\to\infty} f(x) = -3 \quad \text{(horizontal asymptote to the right)}$$
$$\lim_{x\to-\infty} f(x) = 2 \quad \text{(horizontal asymptote to the left)}$$

Note:

$\lim_{x\to\infty} f(x) = L \iff y = L$ horizontal asymptote for $x \to \infty$

$\lim_{x\to-\infty} f(x) = L \iff y = L$ horizontal asymptote for $x \to -\infty$

$\frac{\infty}{\infty}$-type limits (Example: $\lim_{x\to\infty} \frac{x+2}{x-3}$)

Basic Trick for evaluating $\frac{\infty}{\infty}$-type limits (without doing any graphing!):

Divide top and bottom by the largest power of x occurring in the denominator.

Example: a) Evaluate $\lim_{x\to\infty} \dfrac{2x+3}{2-x}$.

Solution: x appears with first degree in denominator.

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x+3}{2-x} &= \lim_{x\to\infty} \frac{(2x+3)\cdot\frac{1}{x}}{(2-x)\cdot\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{2x\cdot\frac{1}{x}+3\cdot\frac{1}{x}}{2\cdot\frac{1}{x}-x\cdot\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{2+\frac{3}{x}}{\frac{2}{x}-1} \\
&= \frac{\lim_{x\to\infty}(2+\frac{3}{x})}{\lim_{x\to\infty}(\frac{2}{x}-1)} = \frac{2+0}{0-1} = -2
\end{aligned}$$

b) What information does the limit in part a) provide about the graph of $f(x) = \dfrac{2x+3}{2-x}$?

Solution:

Horizontal asymptote: $y = -2$ (for $x \to \infty$)

$\lim_{x \to -\infty} \dfrac{2x+3}{2-x} = -2$ (by same method as in part a))

Horizontal asymptote: $y = -2$ (for $x \to -\infty$)

Rational Functions

Example: Evaluate $\lim_{x\to\infty} \dfrac{2x-5x^3}{x^3-x+1}$.

Solution: x appears with 3$^{\text{rd}}$ degree in denominator.

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x-5x^3}{x^3-x+1} &= \lim_{x\to\infty} \frac{(2x-5x^3)\frac{1}{x^3}}{(x^3-x+1)\frac{1}{x^3}} \\
&= \lim_{x\to\infty} \frac{\frac{2x}{x^3}-\frac{5x^3}{x^3}}{\frac{x^3}{x^3}-\frac{x}{x^3}+\frac{1}{x^3}} \\
&= \lim_{x\to\infty} \frac{\frac{2}{x^2}-5}{1-\frac{1}{x^2}+\frac{1}{x^3}} = \frac{0-5}{1-0+0} = -5
\end{aligned}$$

Alternate way:

This works for any rational function (quotient of polynomials)

$$\lim_{x\to\infty} \frac{a\,x^n + \text{ lower degree terms}}{b\,x^m + \text{ lower degree terms}} = \lim_{x\to\infty} \frac{a\,x^n}{b\,x^m}.$$

Drop all lower degree terms.

Example: Redo the limit $\lim_{x\to\infty} \dfrac{2x-5x^3}{x^3-x+1}$ using the alternate way.

Solution:

$$\lim_{x\to\infty} \frac{2x-5x^3}{x^3-x+1} = \lim_{x\to\infty} \frac{-5x^3 \overbrace{+2x}^{\text{lower}}}{x^3 \underbrace{-x+1}_{\text{lower}}}$$

$$= \lim_{x\to\infty} \frac{-5x^3}{x^3}$$

$$= \lim_{x\to\infty} \frac{-5}{1} = -5 \qquad \text{(Books method)}$$

Example: Evaluate $\lim_{x\to\infty} \dfrac{5x^4 - 2x}{6x^3 + 7x^5 - 3}$.

Solution:

First method: x appears with 5^{th} degree in denominator.

$$= \lim_{x\to\infty} \frac{(5x^4 - 2x)\frac{1}{x^5}}{(6x^3 + 7x^5 - 3)\frac{1}{x^5}}$$

$$= \lim_{x\to\infty} \frac{\frac{5}{x} - \frac{2}{x^4}}{\frac{6}{x^2} + 7 - \frac{3}{x^5}}$$

$$= \frac{0 - 0}{0 + 7 - 0} = \frac{0}{7} = 0$$

Second method:

$$= \lim_{x\to\infty} \frac{5x^4}{7x^5}$$

$$= \lim_{x\to\infty} \frac{5}{7x} = 0$$

$\frac{\infty}{\infty}$ type limits with radicals

Example: Compute $\lim_{x\to-\infty} \dfrac{\sqrt{4x^2-2}}{x+3}$.

Solution:

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2-2}}{x+3} = \lim_{x\to-\infty} \frac{\sqrt{4x^2-2}\cdot\frac{1}{x}}{(x+3)\cdot\frac{1}{x}}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{x^2(4-\frac{2}{x^2})}\cdot\frac{1}{x}}{1+\frac{3}{x}}$$

$\sqrt{x^2} = x$?? One has: $\sqrt{x^2} = x$ if $x > 0$ and $\sqrt{x^2} = -x$ if $x < 0$.

$$= \lim_{x\to-\infty} \frac{\overbrace{-x}^{\text{because } x<0}\sqrt{4-\frac{2}{x^2}}\cdot\frac{1}{x}}{1+\frac{3}{x}}$$

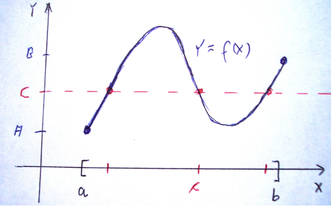
$$= \lim_{x\to-\infty} \frac{-\sqrt{4-\frac{2}{x^2}}}{1+\frac{3}{x}}$$

$$= \frac{-\sqrt{4-0}}{1+0} = \frac{-2}{1} = -2$$

Section 2.8 – Intermediate Value Theorem

Theorem (Intermediate Value Theorem (IVT))

Let $f(x)$ be **continuous** *on the interval $[a, b]$ with $f(a) = A$ and $f(b) = B$.*



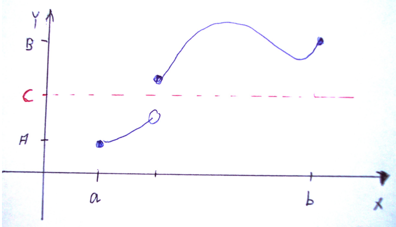
Given any value C between A and B, there is at least one point $c \in [a, b]$ with $f(c) = C$.

Example: Show that $f(x) = x^2$ takes on the value 8 for some x between 2 and 3.

Solution: One has $f(2) = 4$ and $f(3) = 9$. Also $4 < 8 < 9$. Since $f(x)$ is continuous, by the IVT there is a point c, with $2 < c < 3$ with $f(c) = 8$.

Note: The IVT fails if $f(x)$ is not continuous on $[a, b]$.

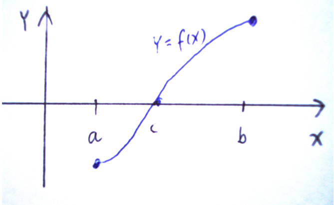
Example:



There is no $c \in [a, b]$ with $f(c) = C$.

Important special case of the IVT:

Suppose that $f(x)$ is **continuous** on the interval $[a, b]$ with $f(a) < 0$ and $f(b) > 0$.



Then there is a point $c \in [a, b]$ where $f(c) = 0$.

Example: Show that the equation

$$x^3 - 3\,x^2 + 1 = 0$$

has a solution on the interval $(0, 1)$.

Solution:

1) $f(x) = x^3 - 3x^2 + 1$ is continuous on $[0, 1]$ (polynomial).

2) $f(0) = 1 > 0$.

3) $f(1) = 1 - 3 + 1 = -1 < 0$.

Thus by the IVT, there is a $c \in (0, 1)$ with $f(c) = 0$.