Calculus I - Lecture 4 - Limits C

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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Section 2.5 — Calculating Limits Algebraically

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1. **Plug-in Types**

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2. $\frac{0}{0}$ **type Limits**

   These are the most important types in Calculus.

   Recall, $\frac{0}{0}$ is an undefined quantity, so plug-in fails.

Example: (Plug-in Type)

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Solution:

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The function $f(x) = \frac{\sqrt{3x^2-1}}{\sin^2 x - \cos(2x)}$ is defined and continuous near $x = \pi$ and so

$$\lim_{x\to\pi} f(x) = f(\pi).$$

Example: ($\frac{0}{0}$ type)

Evaluate $\lim_{x \to 2} \frac{2x^2 - 4x}{x - 2}$.

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$$= \lim_{x\to 2} \frac{2x}{1} \qquad (x \neq 2, \text{ so } (x-2) \text{ cancels out})$$

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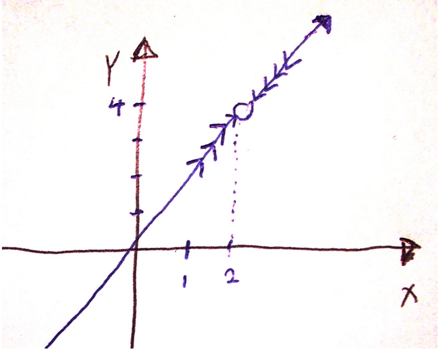
$$= \frac{2 \cdot 2}{1} = 4.$$

Graphical interpretation of the limit in previous example

$y = \frac{2x^2-4x}{x-2} = 2x$ (for $x \neq 2$)



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Trick 1 we did in the previous example.

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$$= \lim_{x\to 3} \frac{-1}{3x} = -\frac{1}{9}.$$

Note: $\frac{0}{0}$ type limits can come out to equal any number or D.N.E. (e.g. infinite limits $\pm\infty$ or left-hand and ride-hand limits are different).

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Recall: The conjugate of $A + \sqrt{B}$ is $A - \sqrt{B}$.

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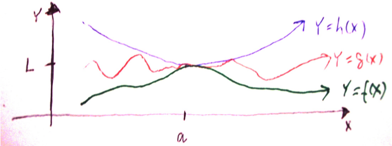
$$= \frac{1}{\sqrt{3+1}+2} = \frac{1}{2+2} = \frac{1}{4}$$

Section 2.6 — Squeeze Theorem and Trigonometric Limits

Theorem (Squeeze Theorem)

Suppose $f(x) \leq g(x) \leq h(h)$ *on an interval containing a and that*

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x).$$



Conclusion: $\lim_{x\to a} g(x) = L.$

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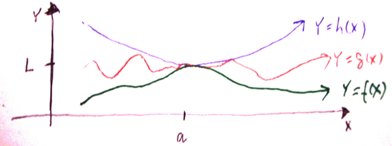

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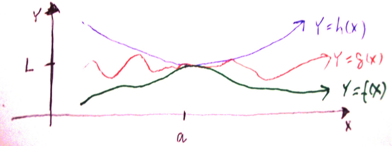

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$$-x \leq x\sin(\tfrac{1}{x}) \leq x \quad (x > 0)$$

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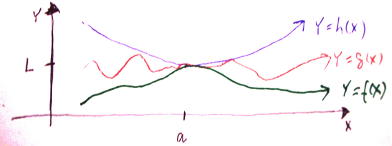

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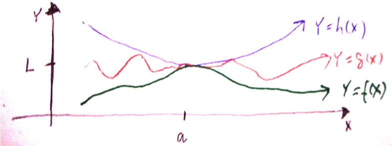

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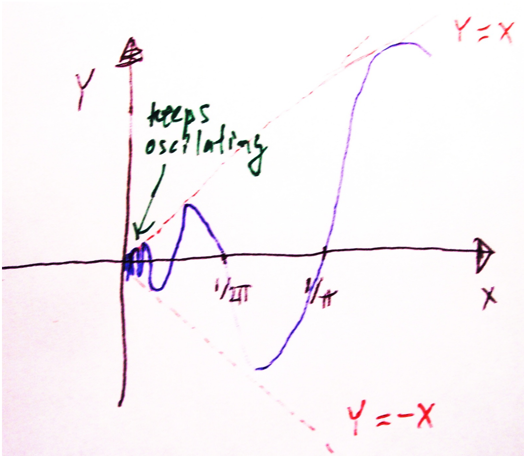
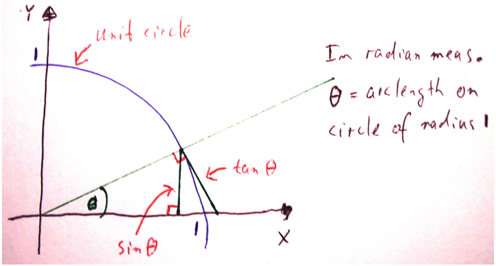
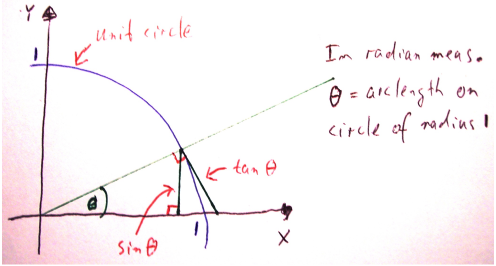
Thus, by the Squeeze Theorem, $\lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right) = 0$


Y = X
Y
keeps oscilating
1/2π
1/π
X
Y = −X

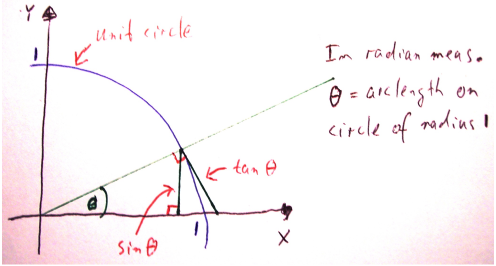


From the diagram: $\sin\theta < \theta < \tan\theta$



From the diagram: $\sin\theta < \theta < \tan\theta$

equivalent to: $\frac{\sin\theta}{\theta} < 1$ and $\theta < \frac{\sin\theta}{\cos\theta}$, so $\frac{\sin\theta}{\theta} > \cos\theta$



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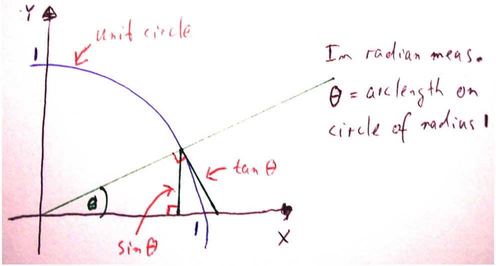
Thus: $$\cos\theta < \frac{\sin\theta}{\theta} < 1$$ ("sandwich")



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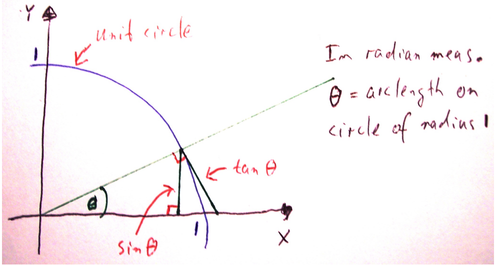
Apply Squeeze Theorem: $\lim_{\theta\to 0}\cos\theta = \cos(0) = 1$

$\lim_{\theta\to 0} 1 = 1$

and we deduce:



From the diagram: $\sin\theta < \theta < \tan\theta$

equivalent to: $\frac{\sin\theta}{\theta} < 1$ and $\theta < \frac{\sin\theta}{\cos\theta}$, so $\frac{\sin\theta}{\theta} > \cos\theta$

Thus: $\cos\theta < \dfrac{\sin\theta}{\theta} < 1$ ("sandwich")

Apply Squeeze Theorem: $\lim_{\theta\to 0}\cos\theta = \cos(0) = 1$

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and we deduce:

Theorem (Basic Trigonometric Limit)

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

Example: Use the basic trigonometric limit to evaluate the following limits.

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