Calculus I - Lecture 3 - Limits B

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

Gerald Hoehn (based on notes by T. Cochran)

January 29, 2014

Section 2.3 — Limit Laws

Last lecture, we learned how to evaluate a limit graphically and numerically.

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4. properties of trigonometric functions

Theorem (Basic Limit Laws)

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(6) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ (*if* $\lim_{x \to a} f(x) \geq 0$ *when* n *even*)

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&= 3 \lim_{x \to 1} x^2 - 5 \lim_{x \to 1} x \quad \text{by (2)}
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&= \sqrt{\frac{3^2+11}{3-1}} = \sqrt{\frac{20}{2}} = \sqrt{10}
\end{aligned}
$$

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Example: Assuming $\lim_{x \to 0} \dfrac{f(x)}{x} = 1$ compute $\lim_{x \to 0} \dfrac{x^2 + 3}{x} \cdot \dfrac{f(x)}{1 - x}$.

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Solution:

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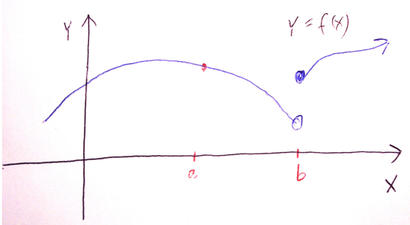
$$= 1 \cdot \frac{\lim_{x\to 0}(x^2+3)}{\lim_{x\to 0}(1-x)} = 1 \cdot \frac{0+3}{1-0} = 3$$

Section 2.4 — Continuity

Intuitive Idea: $f(x)$ is **continuous** at $x = a$ if the graph of $f(x)$ is connected at $x = a$.



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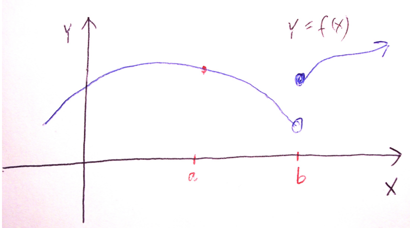

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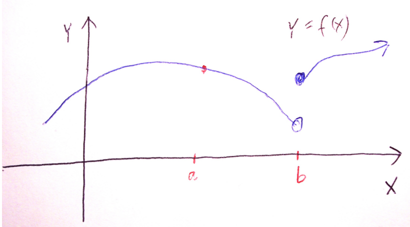

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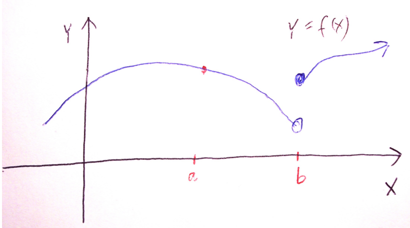

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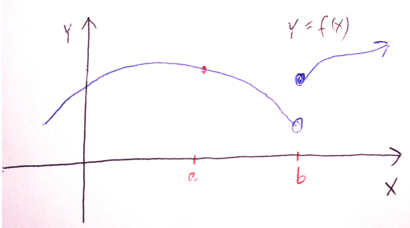

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3. $\lim_{x \to a} f(x) = f(a)$.

In other words, the limit can be evaluated by plugging in $x = a$.

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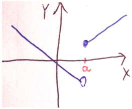
1. Jump: 

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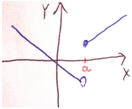
1. Jump: 

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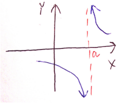
2. Infinite: 

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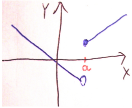
1. Jump: 

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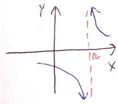
2. Infinite: 

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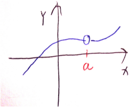
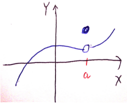
3. Removable: 



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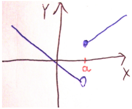
1. Jump: 

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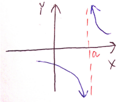
2. Infinite: 

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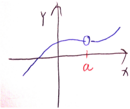
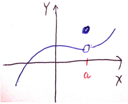
3. Removable: 



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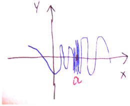
4. Oscillating: 

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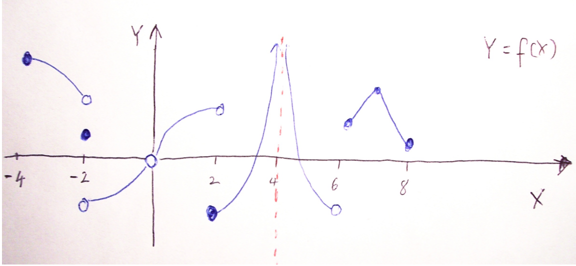

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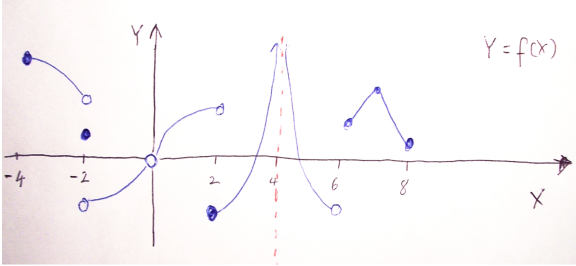


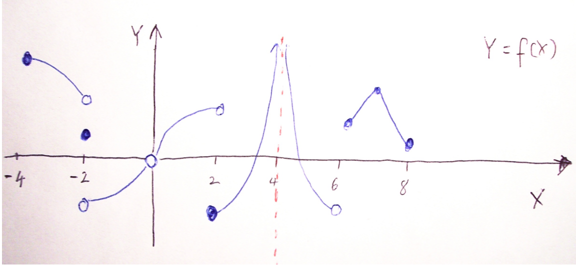
Solution:

a) Discontinuities:

- $x = -2$ (jump)
- $x = 0$ (not defined)
- $x = 2$ (jump)
- $x = 4$ (infinite)
- $x = 6$ (jump)

Example: a) Find the discontinuities of $f(x)$.

b) State the intervals where $f(x)$ is continuous.



Solution:

a) Discontinuities:

- $x = -2$ (jump)
- $x = 0$ (not defined)
- $x = 2$ (jump)
- $x = 4$ (infinite)
- $x = 6$ (jump)

b) Domain of $f(x) \subseteq [-4, 8]$. Intervals where $f(x)$ is continuous:

$[-4, -2)$, $(-2, 0)$, $(0, 2)$, $(2, 4)$, $(4, 6)$, $(6, 8]$.

One sided continuity:

1. $f(x)$ is **continuous from the right** at $x = a$ if
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c) Is $f(x)$ left continuous at $x = 0$?

No: $f(0)$ is not defined.

Example: Determine whether $f(x) = \begin{cases} x^2 - 2, & \text{if } x > 2 \\ 3 - x, & \text{if } x \leq 2 \end{cases}$ is
a) right continuous at $x = 2$, b) left continuous at $x = 2$,
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Solution:

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a) $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 - 2) = 2^2 - 2 = 2.$
$f(2) = 3 - 2 = 1.$
Since $\lim_{x \to 2^+} f(x) \neq f(2)$, $f(x)$ **is not** right continuous at $x = 2$.

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$f(2) = 3 - 2 = 1.$

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c) Since $\lim_{x \to 2^+} f(x) \neq \lim_{x \to 2^-} f(x)$, $f(x)$ **is not** continuous at $x = 2$.

Example: Determine the value(s) of c so that $f(x)$ is continuous at $x = 2$ where $f(x) = \begin{cases} x^2 - 3, & x \geq 2, \\ 2x - c, & x < 2. \end{cases}$

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For $f(x)$ to be continuous we need

$$\begin{aligned} & \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = f(2) \\ \Leftrightarrow \quad & 1 = 4 - c = 1 \\ \Leftrightarrow \quad & c = 4 - 1 = 3. \end{aligned}$$

Theorem

The following functions are continuous:

- $f(x) = x^n$ *for integer numbers* n *everywhere on its domain;*
- $f(x) = x^{1/n}$ *for natural numbers* n *everywhere on its domain;*
- $f(x) = b^x$ *for* $b > 0$ *on the whole real line;*
- $f(x) = \log_b x$ *for* $b > 0$ *and* $x > 0$*;*
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Theorem

Let $f(x)$ *and* $g(x)$ *be continuous at* $x = c$*. Then also the following functions are continuous at* $x = c$*:*

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Theorem

If $g(x)$ *is continuous at* $x = c$ *and* $f(x)$ *is continuous at* $x = g(c)$*, then the composite function* $F(x) = f(g(x))$ *is continuous at* $x = c$*.*

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Since $f(x)$ is continuous at $x=0$ we have

$$\lim_{x->0} f(x) = f(0) = \frac{\cos(0^2)}{1-0^2} = \frac{\cos(0)}{1} = 1.$$