Calculus I - Lecture 27
Volume of Bodies of Revolution

Lecture Notes:
http://www.math.ksu.edu/˜gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

Gerald Hoehn (based on notes by T. Cochran)

February 5, 2014

Section 6.3 – Volumes of Revolution

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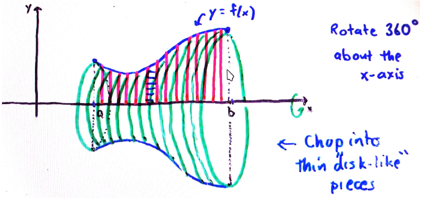

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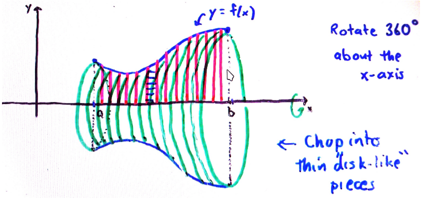

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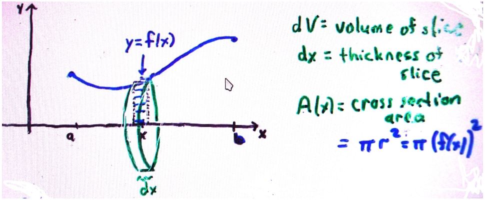

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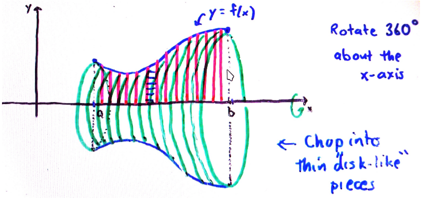

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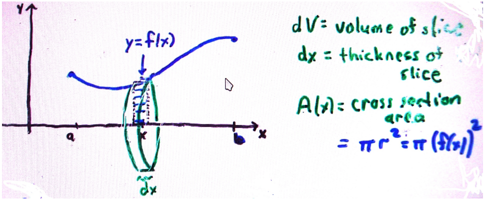

$$V = \int dV = \int_a^b A(x)\,dx = \int_a^b \pi f^2(x)\,dx.$$

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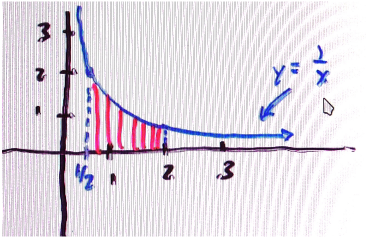

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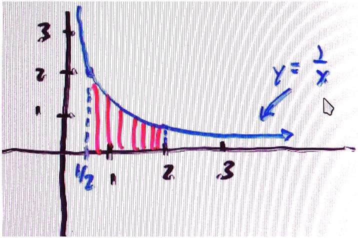

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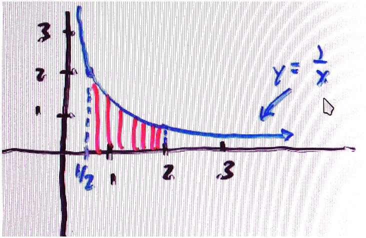

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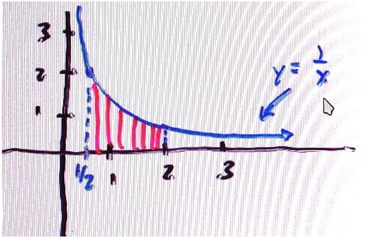

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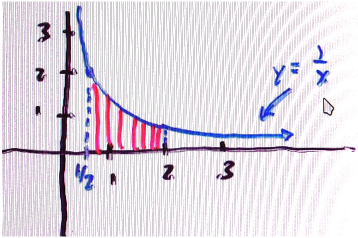

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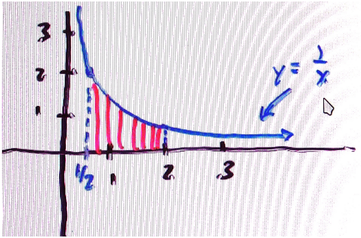

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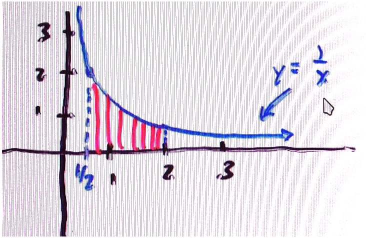

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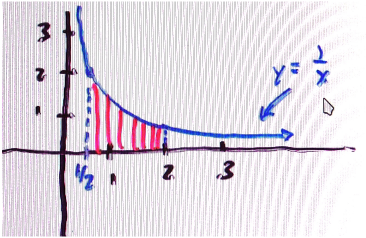

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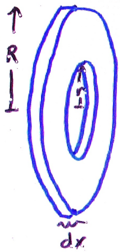

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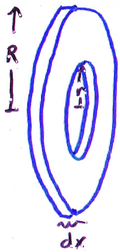

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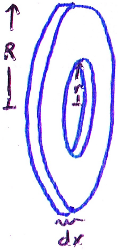

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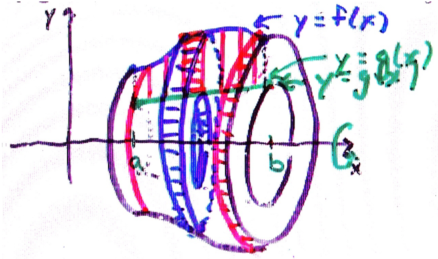

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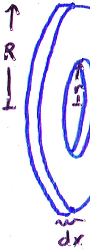

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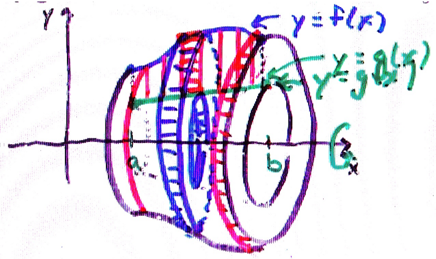

$$V = \int_a^b A\,dx = \int_a^b \pi\left(R^2 - r^2\right)dx = \int_a^b \pi\left(f(x)^2 - g(x)^2\right)dx.$$

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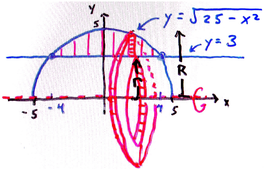

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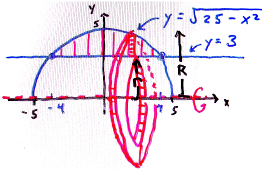

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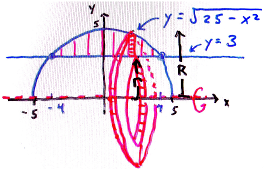

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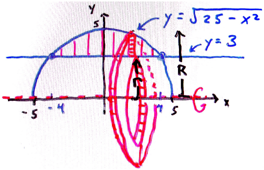

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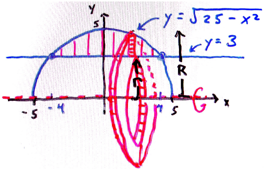

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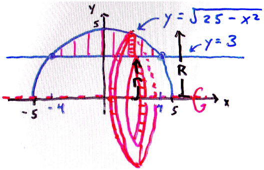

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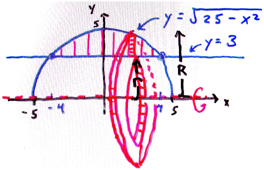

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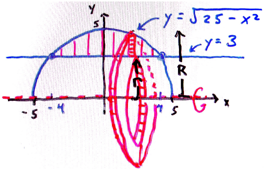

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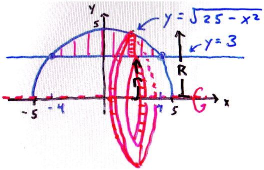

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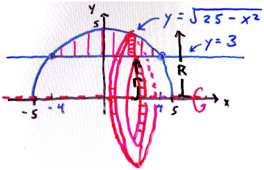

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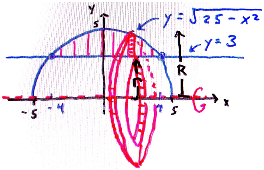

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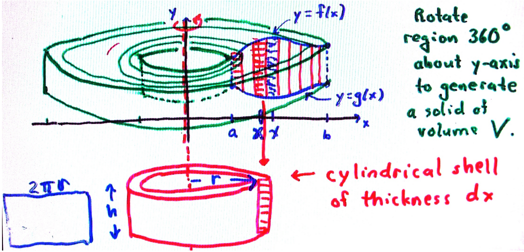
$$= \pi \cdot 2 \cdot \frac{2}{3} \cdot 64 = \frac{256}{3}\pi$$

Section 6.4 – Volumes by Cylindrical Shells



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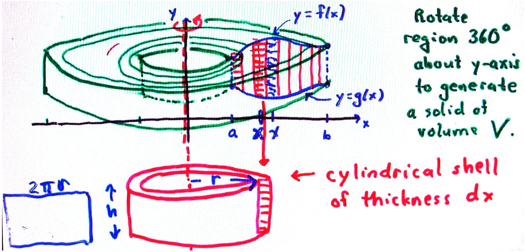

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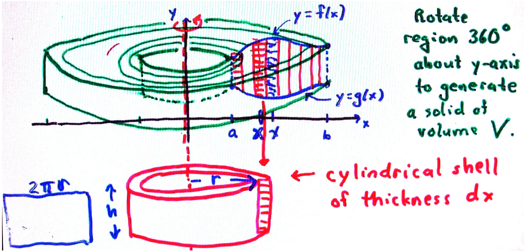

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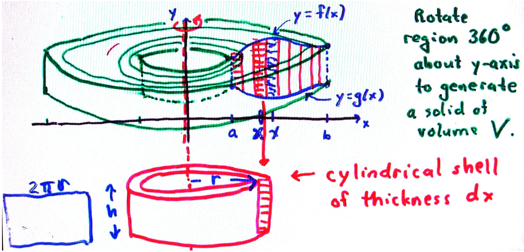

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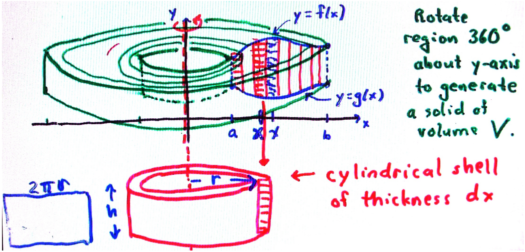

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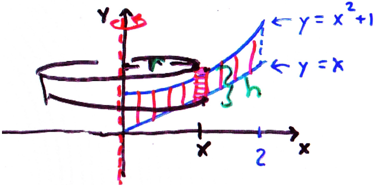
Example: Rotate the region below about the y-axis and determine the volume of the resulting body.



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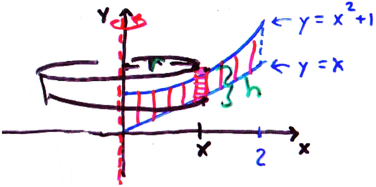

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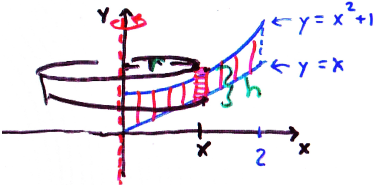

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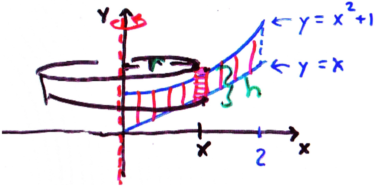
Example: Rotate the region below about the y-axis and determine the volume of the resulting body.



Solution:

$$dV = 2\pi r h \, dx = 2\pi x \big((x^2 + 1) - x \big) \, dx$$

$$V = \int_0^2 2\pi x (x^2 + 1 - x) \, dx$$

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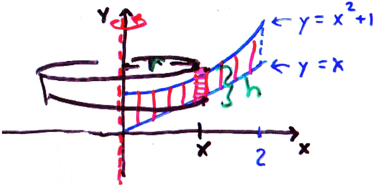

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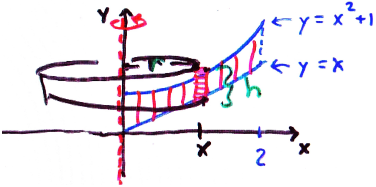

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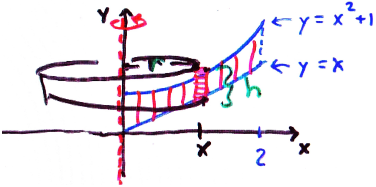
$$= 2\pi \int_0^2 (x^3 - x^2 + x)\, dx$$

$$= 2\pi\Big(\frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2}\Big)\Big|_0^2$$

Example: Rotate the region below about the y-axis and determine the volume of the resulting body.



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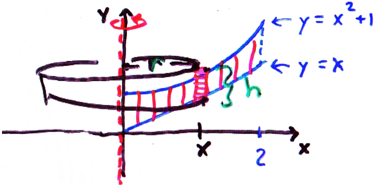
$$V = \int_0^2 2\pi x(x^2 + 1 - x)\, dx$$

$$= 2\pi \int_0^2 (x^3 - x^2 + x)\, dx$$

$$= 2\pi \left(\frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_0^2 = 2\pi \left(4 - \frac{8}{3} + 2\right)$$

Example: Rotate the region below about the y-axis and determine the volume of the resulting body.



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$$dV = 2\pi r h \, dx = 2\pi x\big((x^2+1) - x\big)\, dx$$

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$$= 2\pi \int_0^2 (x^3 - x^2 + x)\, dx$$

$$= 2\pi\left(\frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_0^2 = 2\pi\left(4 - \frac{8}{3} + 2\right) = \frac{20}{3}\pi$$

Example: Rotate the region bounded by $y = x^2$ and $y = 1$ about the the axis $y = -2$ and set up the integral for the volume of the resulting body by a) using washers and b) using shells.

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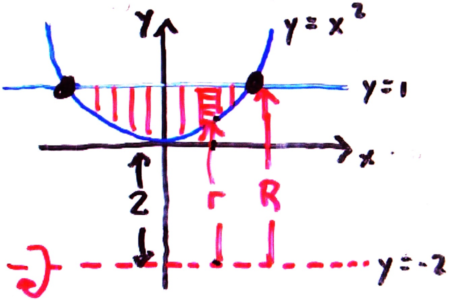
Solution: a)



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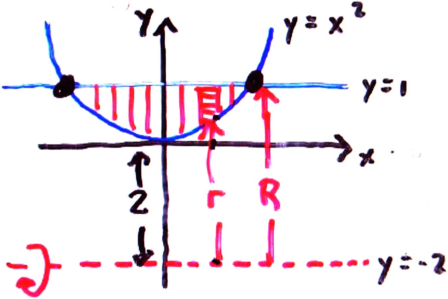

$dV = \text{washer volume} = \pi(R^2 - r^2)\,dx$

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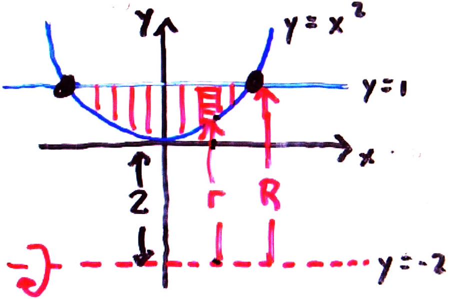

$$dV = \text{washer volume} = \pi(R^2 - r^2)\,dx = \pi\big(3^2 - (2 + x^2)^2\big)\,dx$$

$$V = \int dV$$

Example: Rotate the region bounded by $y = x^2$ and $y = 1$ about the the axis $y = -2$ and set up the integral for the volume of the resulting body by a) using washers and b) using shells.

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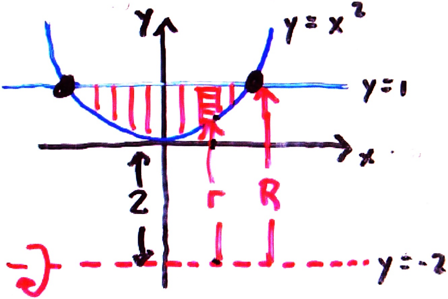
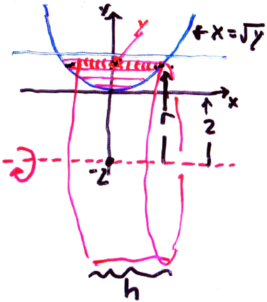


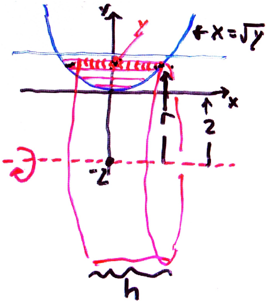
$$dV = \text{washer volume} = \pi(R^2 - r^2)\,dx = \pi\big(3^2 - (2 + x^2)^2\big)\,dx$$

$$V = \int dV = \int_{-1}^{1} \pi\big(9 - (2 + x^2)^2\big)\,dx$$

Solution: b)

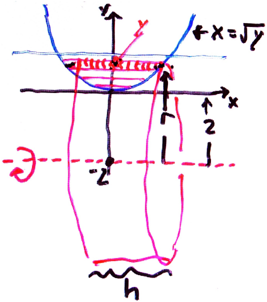


Solution: b)



dV = volume cyl. shell = $2\pi r h\, dy$

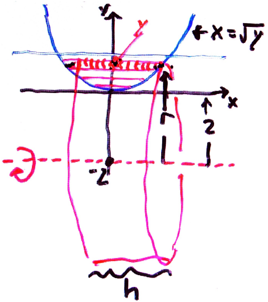
Solution: b)



dV = volume cyl. shell $= 2\pi rh\,dy = 2\pi(2+y)(2\sqrt{y})\,dy$

$V = \int dV$

Solution: b)



dV = volume cyl. shell $= 2\pi r h \, dy = 2\pi(2+y)(2\sqrt{y})\, dy$

$$V = \int dV = \int_0^1 2\pi(2+y)(2\sqrt{y})\, dy$$

Teaching Evaluation

Please use a pencil

Please put on top of chart:

Name of instructor:	Gerald Hoehn
Course:	MATH 220
Course number:	12326
Time:	9:30 a.m.

Next Week: Review by Julie Lang