Calculus I - Lecture 24
The Substitution Method

Lecture Notes:
http://www.math.ksu.edu/˜gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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Integration by Substitution

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Examples from the Table we already know:

$$\int u^n \, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{du}{u} = \ln |u| + C$$

$$\int \sec^2 u \, du = \tan u + C$$

$$\int \frac{du}{1+u^2} = \tan^{-1} u + C$$

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We have determined the antiderivative of $\cos(x^3) \cdot 3x^2$.

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$$\left(-\frac{1}{3}(1-x^2)^{3/2} + C\right)' = -\frac{1}{3}\cdot\frac{3}{2}(1-x^2)^{1/2}\cdot 2x = x\sqrt{1-x^2}$$

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&= \frac{1}{4}\int (u-1)u^{1/2}\,du \\
&= \frac{1}{4}\int \left(u^{3/2} - u^{1/2}\right)\,du \\
&= \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] + C
\end{aligned}$$

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Example: Find $\int x\sqrt{2x+1}\,dx$.

Solution:

Let $u = 2x + 1$. Then $du = 2dx$ or $\frac{du}{2} = dx$.

Also $u = 2x + 1 \;\Leftrightarrow\; u - 1 = 2x \;\Leftrightarrow\; x = \frac{u-1}{2}$

$$\begin{aligned}
\int x\sqrt{2x+1}\,dx &= \int \frac{u-1}{2}\sqrt{u}\cdot\frac{du}{2} \\
&= \frac{1}{4}\int (u-1)u^{1/2}\,du \\
&= \frac{1}{4}\int \left(u^{3/2} - u^{1/2}\right)\,du \\
&= \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] + C \\
&= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C
\end{aligned}$$

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Let $u = \ln x$. Then $du = \dfrac{1}{x}\, dx$

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Example: Find $\frac{dx}{x(\ln x)^2}$.

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Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$

$$\begin{aligned}\int \frac{dx}{x(\ln x)^2} &= \int \frac{1}{u^2}\,du \\ &= \int u^{-2}\,du \\ &= \frac{u^{-1}}{-1} + C \\ &= -\frac{1}{u} + C\end{aligned}$$

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Section 5.7 - Miscellaneous Integrals

From our table of derivatives we obtain the following integrals:

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$$\int \frac{1}{x}\,dx = \ln|x| + C$$

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$$\int \frac{dx}{1+x^2} = \arctan x + C = \tan^{-1} x + C$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C = \sin^{-1} x + C$$

$$\int \frac{dx}{|x|\sqrt{x^2-1}} = \mathrm{arcsec} x + C = \sec^{-1} x + C$$

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Memorize!

The last integral will be done in Calculus II

Example: Find $\int \frac{dx}{\sqrt{9-x^2}}$.

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Solution:

One has $9 - x^2 = 9(1 - \frac{1}{9}x^2) = 9(1 - (\frac{x}{3})^2)$.

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Solution:

One has $9 - x^2 = 9(1 - \frac{1}{9}x^2) = 9(1 - (\frac{x}{3})^2)$.

Let $u = \dfrac{x}{3}$. Then $du = \dfrac{1}{3}\,dx$ or $3\,du = dx$.

$$\begin{aligned}
\int \frac{dx}{\sqrt{9-x^2}} &= \int \frac{dx}{\sqrt{9(1-(\frac{x}{3})^2)}} \\
&= \frac{1}{3}\int \frac{dx}{\sqrt{1-(\frac{x}{3})^2}} \\
&= \frac{1}{3}\int \frac{3\,du}{\sqrt{1-u^2}} \\
&= \arcsin u + C \\
&= \arcsin\left(\frac{x}{3}\right) + C
\end{aligned}$$

Example: Evaluate $\displaystyle\int \frac{dx}{9+4x^2}$.

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