Calculus I - Lecture 22 - The Definite Integral

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

Gerald Hoehn (based on notes by T. Cochran)

February 5, 2014

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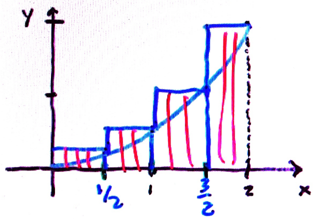

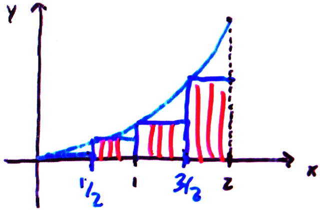

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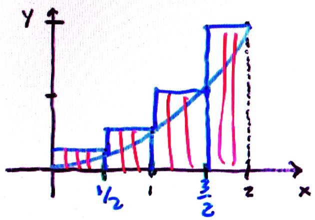

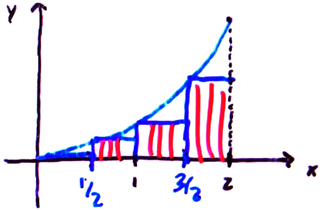

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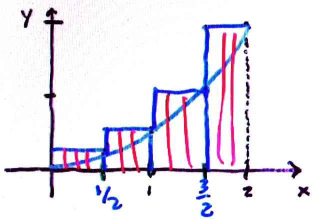

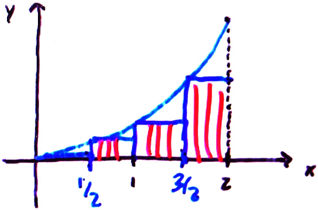

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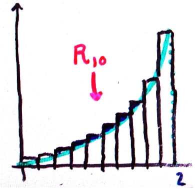

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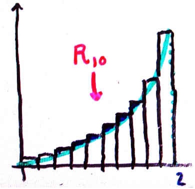

n	R_n	L_n
4	3.75	1.75
10	3.08	2.28
100	2.7068	2.627
1000	2.6707	2.6627
10000	2.6671	2.6663
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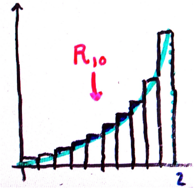

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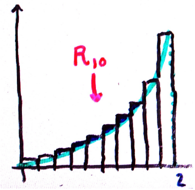

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We'll see a simple way of calculating this next time using antiderivatives!

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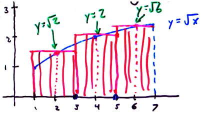

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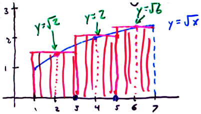

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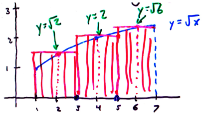

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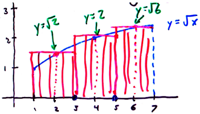

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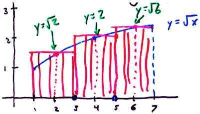

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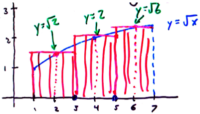

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$s = s(t) =$ position
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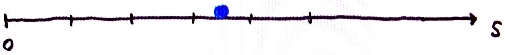

Estimate the distance the object travels during the time interval $[0, 6]$ if the velocity is given by

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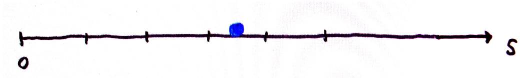

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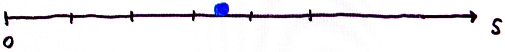

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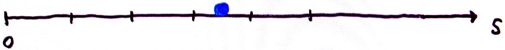

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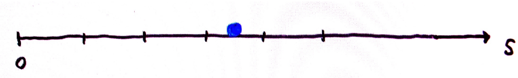

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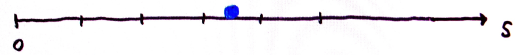

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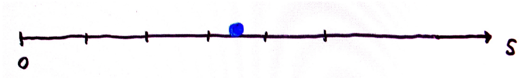

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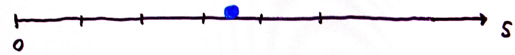

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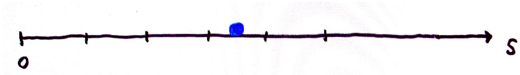

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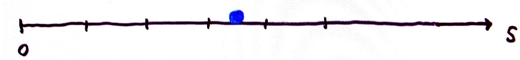

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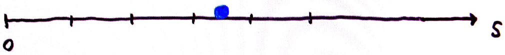

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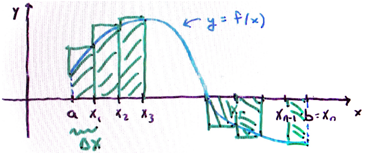

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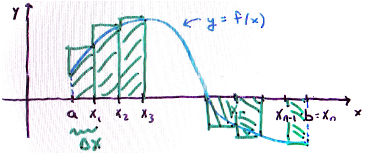

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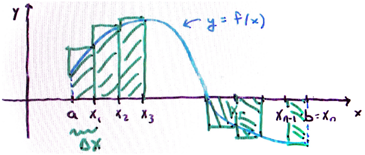

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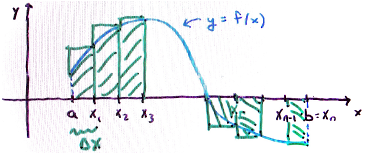

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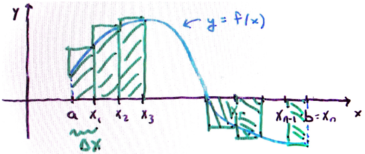

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$$= f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \cdots + f(x_r) \cdot \Delta x + \cdots + f(x_n) \cdot \Delta x$$

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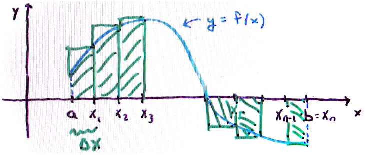

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Let $x_i = a + i \cdot \Delta x$ = right end-point of the i^{th} piece.

$$\begin{aligned} R_n &= \text{Riemann Sum using right end-points} \\ &= f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + \cdots + f(x_r)\cdot\Delta x + \cdots + f(x_n)\cdot\Delta x \\ &= \underbrace{\sum_{i=1}^{n}}_{\text{sum as } i \text{ goes from 1 to } n} \underbrace{f(x_i)\cdot\Delta x}_{i\text{-th term in sum}} \end{aligned}$$

Section 5.2 - Riemann Sums and Definite Integrals

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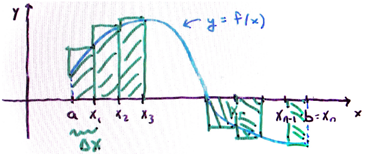

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&= \underbrace{\sum_{i=1}^{n}}_{\substack{\text{sum as } i \text{ goes} \\ \text{from 1 to } n}} \underbrace{f(x_i)\cdot\Delta x}_{i\text{-th term in sum}} \\
&\approx \text{Area above } x\text{-axis bounded by } f(x) \text{ (where } f(x) \text{ is positive)} \\
&\quad - \text{Area below } x\text{-axis bounded by } f(x) \text{ (where } f(x) \text{ is negative)}
\end{aligned}
$$

Definition

The **definite integral** *of* $f(x)$ *over the interval* $[a,b]$ *is given by*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i)\cdot\Delta x\right).$$

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Theorem: The definite integral equals

the area above x-axis bounded by $f(x)$ over $[a,b]$
- the area below x-axis bounded by $f(x)$ over $[a,b]$

also called the “signed” area.

Example: Evaluate $\int_0^5 f(x)\,dx$ for the function below, using the signed area interpretation.

$$f(x) = \begin{cases} x, & 0 \leq x \leq 2 \\ 4 - x, & x > 2. \end{cases}$$

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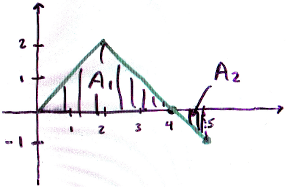

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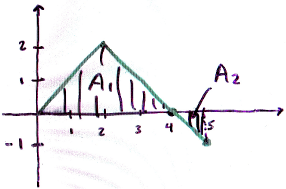

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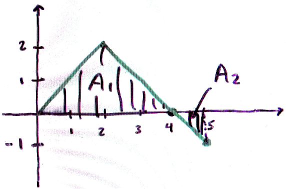

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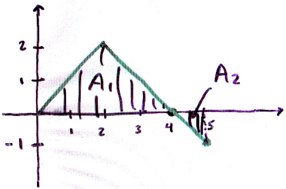

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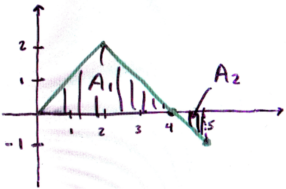

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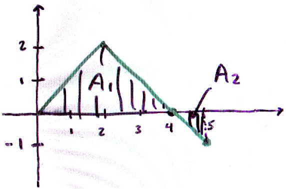

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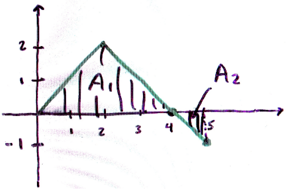

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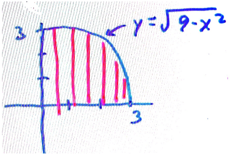

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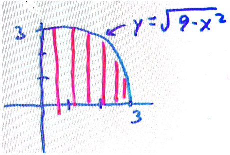

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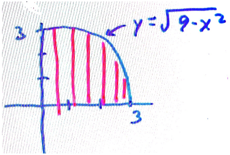

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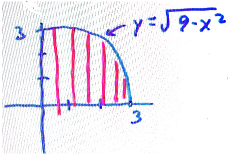

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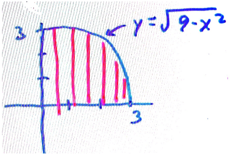

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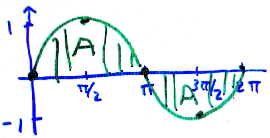

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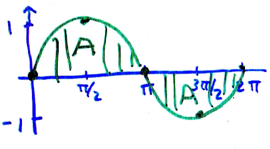

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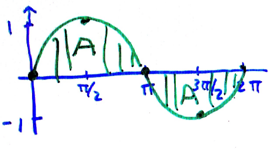

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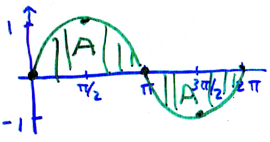

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Suppose $a < b < c$. Then

$$\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$$

Example: Given that $\int_0^a x^2\,dx = \frac{a^3}{3}$, for any $a > 0$, find:

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