Calculus I - Lecture 21 - Review Exam 3

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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April 9, 2014

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y' &= x^x(x^2+1)^{5/2}\left[\ln x + 1 + \frac{5x}{x^2+1}\right]
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Tangent line: $y - 1 = -\frac{2}{5}(x-2)$ or $y = -\frac{2}{5}x + \frac{9}{5}$

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$$\sqrt[3]{27.1} = f(27.1) \approx L(27.1) = 3 + \frac{1}{27}(27.1. - 27) = 3 + \frac{1}{270}.$$

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$$= \frac{80\pi}{3}\ \mathrm{ft}^3/\mathrm{sec}.$$

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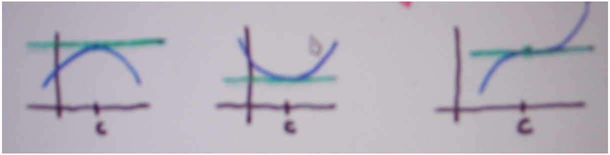

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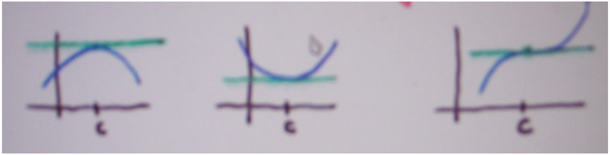
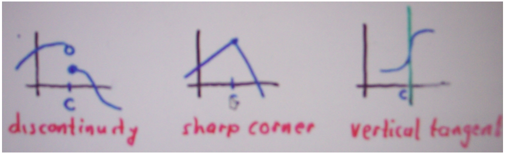


2. *or $f'(c)$ does* **not** *exist.*



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Suppose that $f(x)$ is continuous on the closed interval $[a, b]$. Then $f(x)$ attains its absolute maximum and minimum values on $[a, b]$ at either:

- *A critical point*
- *or one of the end points a or b.*

Example:

a) Find the critical points for the function $f(x) = 3x - x^3$.

b) Find the maximal and minimal values of $f(x) = 3x - x^3$ on the interval $[-1, 3]$.

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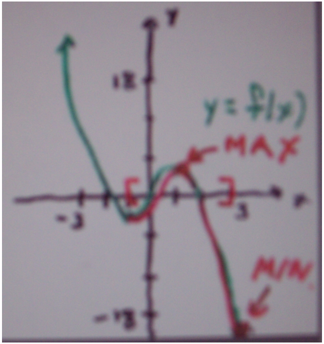
Maximal value is 2 at $x = 1$,
Minimal value is -18 at $x = 3$.


y
12
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-3
3
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-12

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First derivative test:

Let $f'(c) = 0$. Then:

$f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c \Rightarrow f(c)$ is local max.
$f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c \Rightarrow f(c)$ is local min.

Concavity:

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Second derivative test:

Let $f'(c) = 0$. Then:

$f''(c) < 0 \Rightarrow f(c)$ is a local maximum
$f''(c) > 0 \Rightarrow f(c)$ is a local minimum

Concavity:

$f''(x) > 0 \Rightarrow f(x)$ concave up: $f'(x)$ increasing:
$f''(x) < 0 \Rightarrow f(x)$ concave down: $f'(x)$ decreasing:

Inflection points: Points $(x, f(x))$ where the graph of $f(x)$ changes its concavity.

Inflection point test:

Let $f''(c) = 0$.
If $f''(x)$ changes its sign at $x = c$ then $f(x)$ has a inflection point at $x = c$.

Second derivative test:

Let $f'(c) = 0$. Then:

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Transition points: Points where $f'(x)$ or $f''(x)$ has a sign change.

Graph Sketching

Main Steps

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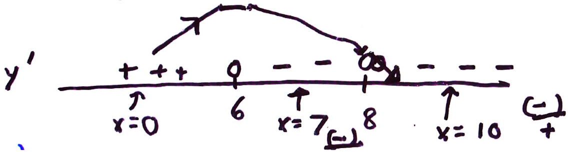

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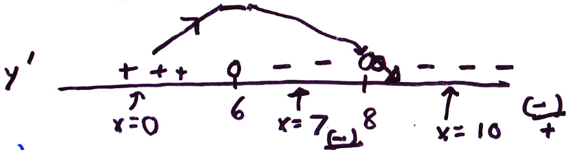

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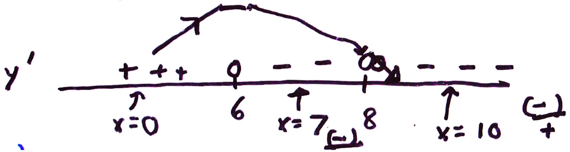

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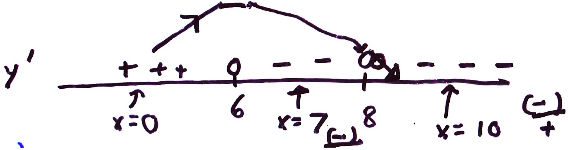

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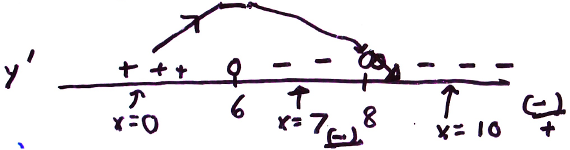

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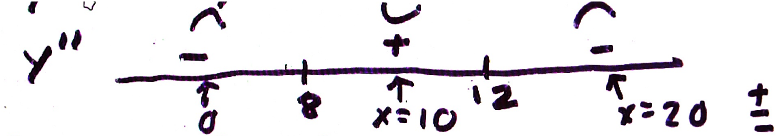

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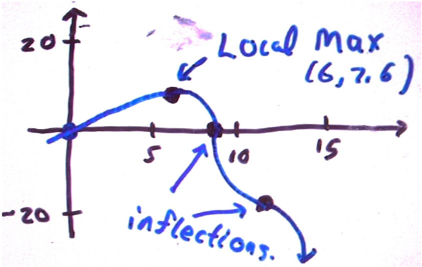

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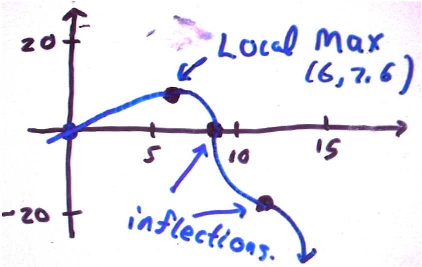

7. No time!

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7. Compute the remaining variables (if asked for) and state the answer in a sentence.

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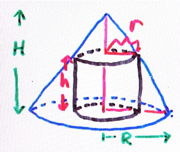

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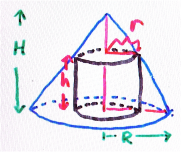

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$h =$ height of cylinder, $r =$radius of cylinder (asked for)
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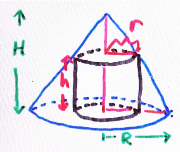

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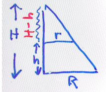

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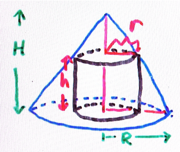

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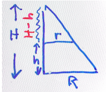

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$$V = \pi \frac{R^2}{H^2}(H-h)^2 h$$

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The volume of the cylinder is maximized for the height $h = \frac{1}{3}H$ and radius $r = \frac{2}{3}R$.