Calculus I - Lecture 20 - The Indefinite Integral

Lecture Notes:

http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:

http://www.math.ksu.edu/math220/spring-2014/indexs14.html

Gerald Hoehn (based on notes by T. Cochran)

April 6, 2014

Reminder: Exam 3 on Thursday, April 10.

Review in Wednesday's Lecture

Practice Exam online on course homepage

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All functions $F(x) = \frac{1}{4}\, x^4 + C$, C any constant, are antiderivatives.

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The indefinite integral or general antiderivative $\int f(x)\mathrm{d}x$ *of a function* $f(x)$ *stands for all possible antiderivatives of* $f(x)$ *defined on an interval, i.e.*

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The indefinite integral should not be confused with the **definite integral** $\int_a^b f(x)\,\mathrm{d}x$ which we will consider next week and is defined as a limit of a sum. The symbol $\int$ is a stretched **S** and reminds about the **S**um. We will also explain the relation between the indefinite and the definite integral.

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e^x	$e^x + C$
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Proof by derivation.

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The guess and fudge method applies to functions of the form $f(ax+b)$, where a and b are constants.

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Since $\int \cos x \mathrm{d}x = \sin x + C$ we try $\sin(3x)$ with fudge factor $\frac{1}{3}$: $\frac{1}{3}\sin(3x)$. Indeed $(\frac{1}{3}\sin(3x))' = \frac{1}{3}\cos(3x)\cdot 3 = \cos(3x)$.
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