Calculus I - Lecture 2 - Limits A

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

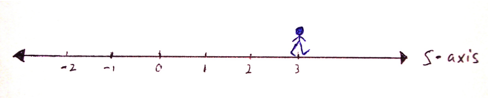
Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

Gerald Hoehn (based on notes by T. Cochran)

January 27, 2014

Section 2.1 — Rates of Change

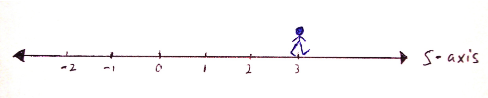
Motion along a straight line:



s = position of object on the line = $s(t)$: function of time
t = time

Section 2.1 — Rates of Change

Motion along a straight line:



s = position of object on the line = $s(t)$: function of time
t = time

$$\text{Average velocity over } [t_1, t_2] = \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{\Delta s}{\Delta t}$$

Δs = change in position,
Δs = change in time

Instantaneous velocity

Fix a time t_0. How fast is the object travelling at this instant?

Instantaneous velocity

Fix a time t_0. How fast is the object travelling at this instant?

Let $v(t_0)$ = instantaneous velocity at t_0,
and v_{ave} = average velocity over the time interval $[t_0, t]$.

$$v(t_0) = \lim_{t \to t_0} v_{\text{ave}} = \lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0}$$

$\lim_{t \to t_0} v_{\text{ave}}$ = the value that v_{ave} approaches as t gets closer and closer to t_0.

Example: A ball falls from a height of 100 ft. Approximate its speed 2 sec. after being relased.

t	0	1	1.5	1.9	1.99	1.999	2
s	100	84	64	42.24	36.638	36.06399	36

$s =$ height (ft.)

$t =$ time (sec.) (observed data)

Example: A ball falls from a height of 100 ft. Approximate its speed 2 sec. after being relased.

t	0	1	1.5	1.9	1.99	1.999	2
s	100	84	64	42.24	36.638	36.06399	36

$s =$ height (ft.)
$t =$ time (sec.) (observed data)

Solution:

time interval	Δt (sec.)	Δs (ft.)	Averarage velocity $\Delta t/\Delta s$ (ft./sec.)
[1,2]	1	$84 - 36 = 48$	$48/1 = 48$
[1.5,2]	.5	$64 - 36 = 28$	$28/.05 = 56$
[1.9,2]	.1	$42.24 - 36 = 6.24$	$6.24/.1 = 62.4$
[1.99,2]	.01	$36.638 - 36 = .638$	$.638/.01 = 63.8$
[1.999,2]	.001	$36.06399 - 36 = .06399$	$.06399/.001 = 63.99$

Example: A ball falls from a height of 100 ft. Approximate its speed 2 sec. after being relased.

t	0	1	1.5	1.9	1.99	1.999	2
s	100	84	64	42.24	36.638	36.06399	36

s = height (ft.)
t = time (sec.) (observed data)

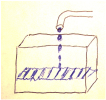
Solution:

time interval	Δt (sec.)	Δs (ft.)	Averarage velocity $\Delta t/\Delta s$ (ft./sec.)
[1,2]	1	$84 - 36 = 48$	$48/1 = 48$
[1.5,2]	.5	$64 - 36 = 28$	$28/.05 = 56$
[1.9,2]	.1	$42.24 - 36 = 6.24$	$6.24/.1 = 62.4$
[1.99,2]	.01	$36.638 - 36 = .638$	$.638/.01 = 63.8$
[1.999,2]	.001	$36.06399 - 36 = .06399$	$.06399/.001 = 63.99$

$$\lim_{t\to 2} \frac{\Delta s}{\Delta t} = \lim_{\Delta t\to 0} \frac{\Delta s}{\Delta t} = 64 \text{ ft./sec.}$$

is the speed as the data is showing.

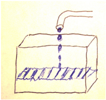
Example: The volume of water in a tank is given by



$$V(t) = 1 - e^{-t} \text{ where}$$

V = Volume in cubic meters,
t = time in seconds.
At what rate is the volume increasing when $t = 5$ sec.?
Estimate numerically!

Example: The volume of water in a tank is given by



$$V(t) = 1 - e^{-t} \text{ where}$$

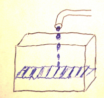
V = Volume in cubic meters,
t = time in seconds.
At what rate is the volume increasing when $t = 5$ sec.?
Estimate numerically!

Solution: First approximation $\Delta t = .1$ sec.

$$\frac{\Delta V}{\Delta t} = \frac{V(5.1) - V(5)}{.1} = \frac{.993903 - .993262}{.1} = .00641 \ \mathrm{m}^3/\text{sec.}$$

Example: The volume of water in a tank is given by

$$V(t) = 1 - e^{-t} \text{ where}$$



V = Volume in cubic meters,
t = time in seconds.
At what rate is the volume increasing when $t = 5$ sec.?
Estimate numerically!

Solution: First approximation $\Delta t = .1$ sec.

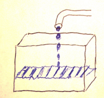
$$\frac{\Delta V}{\Delta t} = \frac{V(5.1) - V(5)}{.1} = \frac{.993903 - .993262}{.1} = .00641 \text{ m}^3/\text{sec.}$$

Improved approximation $\Delta t = .001$ sec.

$$\frac{\Delta V}{\Delta t} = \frac{V(5.001) - V(5)}{.1} = \frac{.99326879 - .99326205}{.001} = .00674 \text{ m}^3/\text{sec.}$$

Example: The volume of water in a tank is given by

$$V(t) = 1 - e^{-t} \text{ where}$$



V = Volume in cubic meters,
t = time in seconds.
At what rate is the volume increasing when $t = 5$ sec.?
Estimate numerically!

Solution: First approximation $\Delta t = .1$ sec.

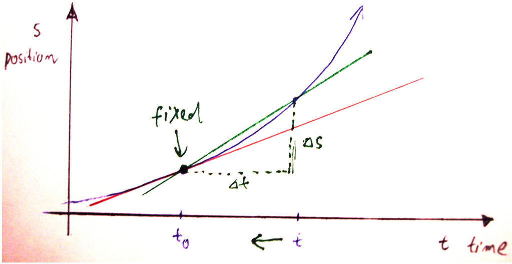
$$\frac{\Delta V}{\Delta t} = \frac{V(5.1) - V(5)}{.1} = \frac{.993903 - .993262}{.1} = .00641 \ \mathrm{m}^3/\text{sec.}$$

Improved approximation $\Delta t = .001$ sec.

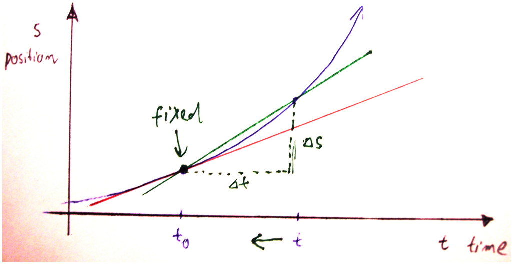
$$\frac{\Delta V}{\Delta t} = \frac{V(5.001) - V(5)}{.1} = \frac{.99326879 - .99326205}{.001} = .00674 \ \mathrm{m}^3/\text{sec.}$$

(Exact value is $e^{-5} = .006737\ldots$ as we'll see later.)

Graphical interpretation of average and instantaneous velocity



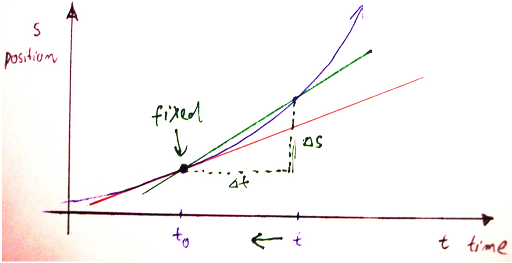
Graphical interpretation of average and instantaneous velocity



$$v_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{s(t) - s(t_0)}{t - t_0}$$

$$= \text{slope of "secant" line to graph,}$$

Graphical interpretation of average and instantaneous velocity



$$v_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{s(t) - s(t_0)}{t - t_0}$$

$= $ slope of "secant" line to graph,

$$v(t_0) = \lim_{t \to t_0} \frac{\Delta s}{\Delta t}$$

$= $ slope of the tangent line to the curve at the point $(t_0, s(t_0))$.

Section 2.2 — The Concept of Limit

$$\lim_{x \to a} f(x) \;=\; \text{“limit as } x \text{ approaches } a \text{ of } f(x)\text{”}$$

$:=$ the value that $f(x)$ approaches as x gets closer and closer to a

Section 2.2 — The Concept of Limit

$$\lim_{x \to a} f(x) \; = \; \text{“limit as } x \text{ approaches } a \text{ of } f(x)\text{”}$$

$:=$ the value that $f(x)$ approaches as x gets closer and closer to a

Notes: 1) x is allowed to approach a from the right or left, **but** x **is not allowed to equal** a**.**

Section 2.2 — The Concept of Limit

$$\lim_{x \to a} f(x) = \text{“limit as } x \text{ approaches } a \text{ of } f(x)\text{”}$$

$:=$ the value that $f(x)$ approaches as x gets closer and closer to a

Notes: 1) x is allowed to approach a from the right or left, **but x is not allowed to equal a.**

2) In order for the limit to exist you must get the **same value** approaching from the left or the right.

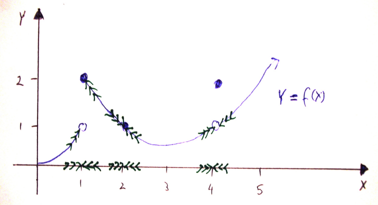
Section 2.2 — The Concept of Limit

$$\lim_{x \to a} f(x) \;=\; \text{“limit as } x \text{ approaches } a \text{ of } f(x)\text{”}$$

$:=$ the value that $f(x)$ approaches as x gets closer and closer to a

Notes: 1) x is allowed to approach a from the right or left, **but x is not allowed to equal a.**

2) In order for the limit to exist you must get the **same value** approaching from the left or the right.



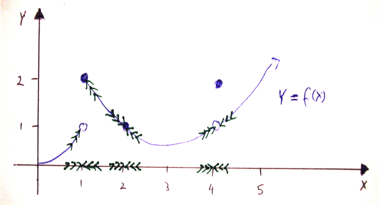
a) $\lim_{x \to 2} f(x) =$

Section 2.2 — The Concept of Limit

$$\lim_{x \to a} f(x) \;=\; \text{"limit as } x \text{ approaches } a \text{ of } f(x)\text{"}$$

$:=$ the value that $f(x)$ approaches as x gets closer and closer to a

Notes: 1) x is allowed to approach a from the right or left, **but x is not allowed to equal a.**
2) In order for the limit to exist you must get the **same value** approaching from the left or the right.



a) $\lim_{x \to 2} f(x) = 1$;

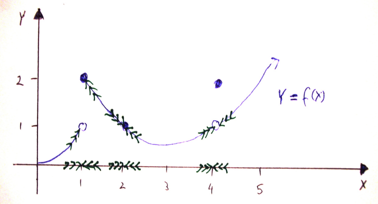
Section 2.2 — The Concept of Limit

$$\lim_{x \to a} f(x) = \text{“limit as } x \text{ approaches } a \text{ of } f(x)\text{”}$$

$:=$ the value that $f(x)$ approaches as x gets closer and closer to a

Notes: 1) x is allowed to approach a from the right or left, **but x is not allowed to equal a.**

2) In order for the limit to exist you must get the **same value** approaching from the left or the right.



a) $\lim_{x \to 2} f(x) = 1$;

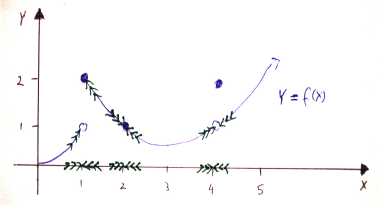
b) $\lim_{x \to 1} f(x) =$

Section 2.2 — The Concept of Limit

$$\lim_{x \to a} f(x) \;=\; \text{“limit as } x \text{ approaches } a \text{ of } f(x)\text{”}$$

$:=$ the value that $f(x)$ approaches as x gets closer and closer to a

Notes: 1) x is allowed to approach a from the right or left, **but** x **is not allowed to equal** a.
2) In order for the limit to exist you must get the **same value** approaching from the left or the right.



a) $\lim_{x \to 2} f(x) = 1$;

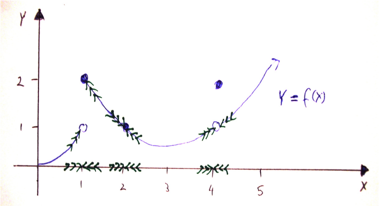
b) $\lim_{x \to 1} f(x) =$ D.N.E. (left and right limits are different);

Section 2.2 — The Concept of Limit

$$\lim_{x \to a} f(x) = \text{“limit as } x \text{ approaches } a \text{ of } f(x)\text{”}$$

$:=$ the value that $f(x)$ approaches as x gets closer and closer to a

Notes: 1) x is allowed to approach a from the right or left, **but** x **is not allowed to equal** a**.**
2) In order for the limit to exist you must get the **same value** approaching from the left or the right.



a) $\lim_{x \to 2} f(x) = 1$;
b) $\lim_{x \to 1} f(x) =$ D.N.E. (left and right limits are different);
c) $\lim_{x \to 4} f(x) =$

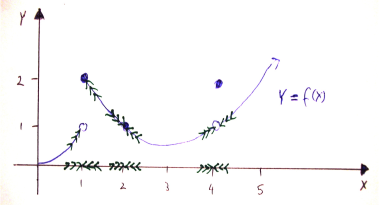
Section 2.2 — The Concept of Limit

$$\lim_{x \to a} f(x) \;=\; \text{“limit as } x \text{ approaches } a \text{ of } f(x)\text{”}$$

$:=$ the value that $f(x)$ approaches as x gets closer and closer to a

Notes: 1) x is allowed to approach a from the right or left, **but x is not allowed to equal a.**

2) In order for the limit to exist you must get the **same value** approaching from the left or the right.

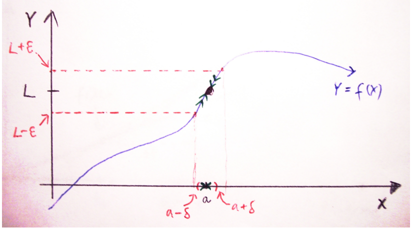


a) $\lim_{x \to 2} f(x) = 1$;

b) $\lim_{x \to 1} f(x) =$ D.N.E. (left and right limits are different);

c) $\lim_{x \to 4} f(x) = 1$ (x never reaches 4).

How to make this definition rigorous?



Definition: Suppose that $f(x)$ is defined on some open interval containing a (but possibly not at a). We say

$$\lim_{x \to a} f(x) = L$$

if for any (tiny) positive number ε (epsilon) we have

$$|f(x) - L| < \varepsilon$$

provided x is sufficiently close to a.

"Sufficiently close" means there exists, a (tiny) positive number δ (delta) such that if $|x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

You will not be asked to do a ε-δ proof in this class.

One-sided limits:

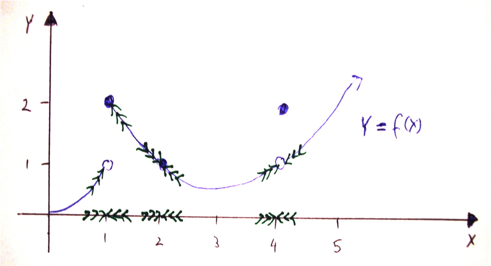
$\lim_{x \to a^+} f(x) =$ "limit of $f(x)$ as x approaches a from the right"

$\lim_{x \to a^-} f(x) =$ "limit of $f(x)$ as x approaches a from the left"

One-sided limits:

$\lim_{x \to a^+} f(x) =$ "limit of $f(x)$ as x approaches a from the right"

$\lim_{x \to a^-} f(x) =$ "limit of $f(x)$ as x approaches a from the left"

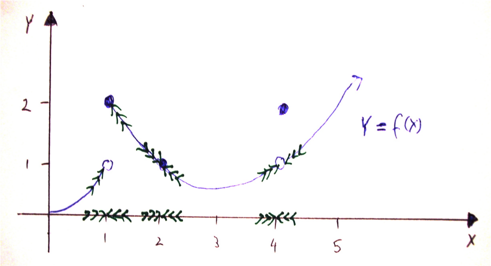


a) $\lim_{x \to 1^-} f(x) =$

One-sided limits:

$$\lim_{x \to a^+} f(x) = \text{"limit of } f(x) \text{ as } x \text{ approaches } a \text{ from the right"}$$

$$\lim_{x \to a^-} f(x) = \text{"limit of } f(x) \text{ as } x \text{ approaches } a \text{ from the left"}$$

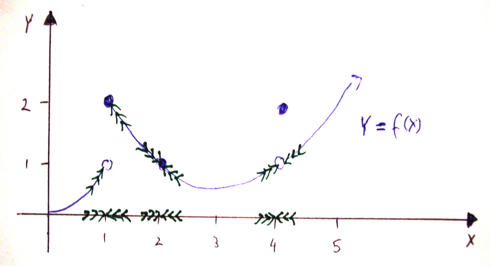


a) $\lim_{x \to 1^-} f(x) = 1 \quad (x < 1);$

One-sided limits:

$\lim_{x \to a^+} f(x) =$ "limit of $f(x)$ as x approaches a from the right"

$\lim_{x \to a^-} f(x) =$ "limit of $f(x)$ as x approaches a from the left"



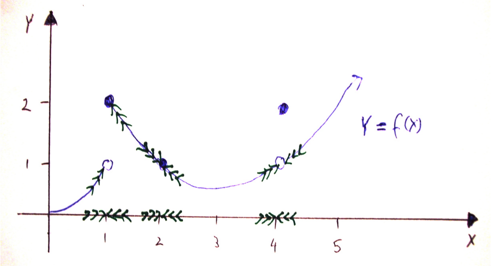
a) $\lim_{x \to 1^-} f(x) = 1 \quad (x < 1);$

b) $\lim_{x \to 1^+} f(x) =$

One-sided limits:

$\lim_{x \to a^+} f(x) =$ "limit of $f(x)$ as x approaches a from the right"

$\lim_{x \to a^-} f(x) =$ "limit of $f(x)$ as x approaches a from the left"



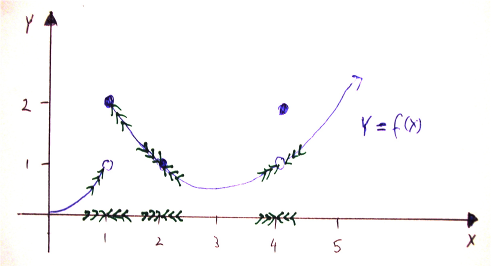
a) $\lim_{x \to 1^-} f(x) = 1 \quad (x < 1);$

b) $\lim_{x \to 1^+} f(x) = 2 \quad (x > 1);$

One-sided limits:

$\lim_{x \to a^+} f(x) =$ "limit of $f(x)$ as x approaches a from the right"

$\lim_{x \to a^-} f(x) =$ "limit of $f(x)$ as x approaches a from the left"



a) $\lim_{x \to 1^-} f(x) = 1 \quad (x < 1)$;

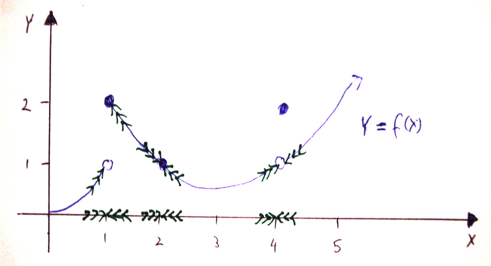
b) $\lim_{x \to 1^+} f(x) = 2 \quad (x > 1)$;

c) $\lim_{x \to 2^-} f(x) =$

One-sided limits:

$\lim_{x \to a^+} f(x) =$ "limit of $f(x)$ as x approaches a from the right"

$\lim_{x \to a^-} f(x) =$ "limit of $f(x)$ as x approaches a from the left"



a) $\lim_{x \to 1^-} f(x) = 1 \quad (x < 1)$;

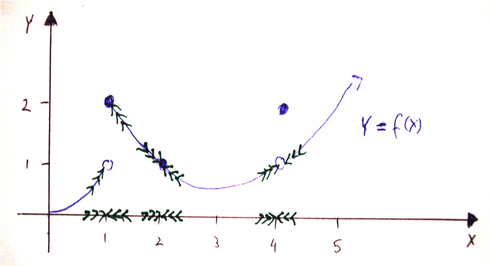
b) $\lim_{x \to 1^+} f(x) = 2 \quad (x > 1)$;

c) $\lim_{x \to 2^-} f(x) = 1 \quad (x < 2)$;

One-sided limits:

$\lim_{x \to a^+} f(x) =$ "limit of $f(x)$ as x approaches a from the right"

$\lim_{x \to a^-} f(x) =$ "limit of $f(x)$ as x approaches a from the left"



a) $\lim_{x \to 1^-} f(x) = 1 \quad (x < 1);$

b) $\lim_{x \to 1^+} f(x) = 2 \quad (x > 1);$

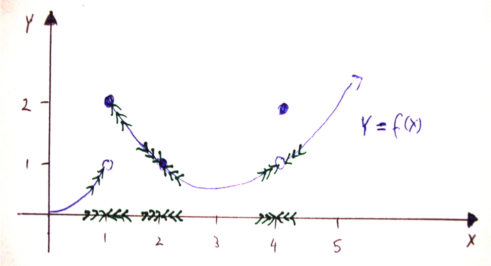
c) $\lim_{x \to 2^-} f(x) = 1 \quad (x < 2);$

d) $\lim_{x \to 4^-} f(x) =$

One-sided limits:

$$\lim_{x \to a^+} f(x) = \text{“limit of } f(x) \text{ as } x \text{ approaches } a \text{ from the right”}$$

$$\lim_{x \to a^-} f(x) = \text{“limit of } f(x) \text{ as } x \text{ approaches } a \text{ from the left”}$$



a) $\lim_{x \to 1^-} f(x) = 1 \quad (x < 1);$

b) $\lim_{x \to 1^+} f(x) = 2 \quad (x > 1);$

c) $\lim_{x \to 2^-} f(x) = 1 \quad (x < 2);$

d) $\lim_{x \to 4^-} f(x) = 1 \quad (x < 4,\ x \text{ never reaches } 4).$

Numerical determination of limits

Example: Evaluate $\lim_{x \to 0^+} (1+x)^{1/x}$.

Numerical determination of limits

Example: Evaluate $\lim_{x \to 0^+} (1+x)^{1/x}$.

Solution: Set up table on calculator

$\boxed{\texttt{y=}}$ $\quad y_1 = (1+x) \wedge (1/x)$

$\boxed{\texttt{2nd}}$ $\boxed{\texttt{TBLSET}}$ $\quad$ Indpnt $\boxed{\texttt{ASK}}$

$\boxed{\texttt{2nd}}$ $\boxed{\texttt{TABLE}}$

Numerical determination of limits

Example: Evaluate $\lim_{x \to 0^+} (1+x)^{1/x}$.

Solution: Set up table on calculator

$\boxed{\texttt{y=}}$ $\quad y_1 = (1+x) \wedge (1/x)$

$\boxed{\texttt{2nd}}$ $\boxed{\texttt{TBLSET}}$ $\quad$ Indpnt $\boxed{\texttt{ASK}}$

$\boxed{\texttt{2nd}}$ $\boxed{\texttt{TABLE}}$

(Since $x \to 0^+$ we evaluate f(x) at small positive values of x approaching 0.)

x	y_1
.1	2.59374
.01	2.70481
.001	2.71692
.0001	2.71814
.00001	2.71827

Numerical determination of limits

Example: Evaluate $\lim_{x \to 0^+} (1+x)^{1/x}$.

Solution: Set up table on calculator

$\boxed{\texttt{y=}}$ $y_1 = (1+x) \wedge (1/x)$

$\boxed{\texttt{2nd}}$ $\boxed{\texttt{TBLSET}}$ Indpnt $\boxed{\texttt{ASK}}$

$\boxed{\texttt{2nd}}$ $\boxed{\texttt{TABLE}}$

(Since $x \to 0^+$ we evaluate f(x) at small positive values of x approaching 0.)

x	y_1
.1	2.59374
.01	2.70481
.001	2.71692
.0001	2.71814
.00001	2.71827

$$\lim_{x \to 0^+} (1+x)^{1/x} = e = 2.7182818\ldots$$

Example: Evaluate the limit $\lim_{x \to 0^+} \sin\left(\frac{1}{x}\right)$ numerically

Example: Evaluate the limit $\lim_{x \to 0^+} \sin\left(\frac{1}{x}\right)$ numerically

Solution:

x	$\sin\left(\frac{1}{x}\right)$
.1	$-.54402$
.01	$-.50636$
.001	.82688
.0001	$-.30561$
.00001	.035748

Example: Evaluate the limit $\lim_{x \longrightarrow 0^+} \sin\left(\frac{1}{x}\right)$ numerically

Solution:

x	$\sin\left(\frac{1}{x}\right)$
.1	$-.54402$
.01	$-.50636$
.001	.82688
.0001	$-.30561$
.00001	.035748

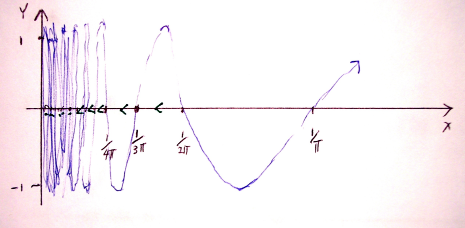
No obvious limit. Let us investigate the graph of $\sin\left(\frac{1}{x}\right)$.

Example: Evaluate the limit $\lim_{x \to 0^+} \sin\left(\frac{1}{x}\right)$ numerically

Solution:

x	$\sin\left(\frac{1}{x}\right)$
.1	$-.54402$
.01	$-.50636$
.001	.82688
.0001	$-.30561$
.00001	.035748

No obvious limit. Let us investigate the graph of $\sin\left(\frac{1}{x}\right)$.
We note that $\sin\left(\frac{1}{x}\right) = 0$ for $\frac{1}{x} = \pi, 2\pi, 3\pi, 4\pi, \ldots$
$\Leftrightarrow x = \frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \frac{1}{4\pi}, \ldots$.
Thus the graph oscillates for $x \to 0^+$ infinitely many times:



$\lim_{x \to 0^+} \sin\left(\frac{1}{x}\right)$ D.N.E.

Example: Investigate the limits numerically:

$$\lim_{x \to 2^-} \frac{2x+3}{x^2-4} \qquad \text{and} \qquad \lim_{x \to 2^+} \frac{2x+3}{x^2-4}$$

Example: Investigate the limits numerically:

$$\lim_{x\to 2^-} \frac{2x+3}{x^2-4} \qquad \text{and} \qquad \lim_{x\to 2^+} \frac{2x+3}{x^2-4}$$

Solution: Let $y = (2x+3)/(x^2-4)$. Make tables:

x	y
1.9	-17.44
1.99	-174.9
1.999	-1750

x	y
2.1	17.56
2.01	175.1
2.001	1750

Example: Investigate the limits numerically:

$$\lim_{x \to 2^-} \frac{2x+3}{x^2-4} \qquad \text{and} \qquad \lim_{x \to 2^+} \frac{2x+3}{x^2-4}$$

Solution: Let $y = (2x+3)/(x^2-4)$. Make tables:

x	y
1.9	−17.44
1.99	−174.9
1.999	−1750

x	y
2.1	17.56
2.01	175.1
2.001	1750

$$\lim_{x \to 2^-} \frac{2x+3}{x^2-4} = -\infty \text{ (D.N.E.)} \qquad \lim_{x \to 2^+} \frac{2x+3}{x^2-4} = +\infty \text{ (D.N.E.)}$$

Example: Investigate the limits numerically:

$$\lim_{x\to 2^-} \frac{2x+3}{x^2-4} \quad \text{and} \quad \lim_{x\to 2^+} \frac{2x+3}{x^2-4}$$

Solution: Let $y = (2x+3)/(x^2-4)$. Make tables:

x	y
1.9	−17.44
1.99	−174.9
1.999	−1750

x	y
2.1	17.56
2.01	175.1
2.001	1750

$$\lim_{x\to 2^-} \frac{2x+3}{x^2-4} = -\infty \text{ (D.N.E.)} \qquad \lim_{x\to 2^+} \frac{2x+3}{x^2-4} = +\infty \text{ (D.N.E.)}$$

Here, the graph has a vertical asymptote: