Calculus I - Lecture 19 - Applied Optimization

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

Gerald Hoehn (based on notes by T. Cochran)

April 2, 2014

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7. Compute the remaining variables (if asked for) and state the answer in a sentence.

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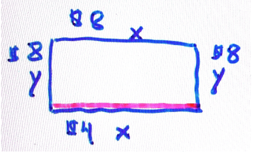
1. 

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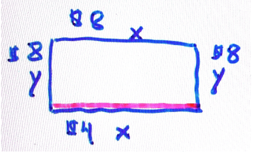
1. 

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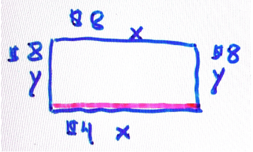
1. 

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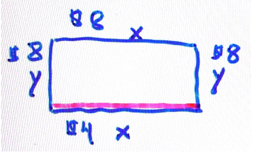
1. 

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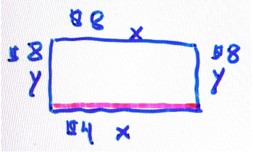
1. 

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$$\frac{dC}{dx} = 12 - 1200x^{-2} = 0 \Leftrightarrow 12 = 1200x^{-2} \Leftrightarrow x^2 = 100 \Leftrightarrow x = 10$$

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The total cost of the fence is minimized by a garden length of 10 feet and a width of 7.5 feet.

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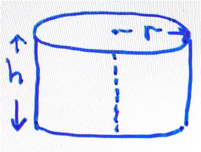

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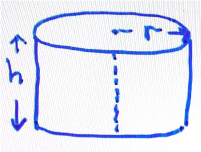

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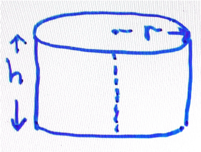

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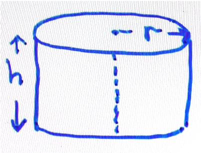

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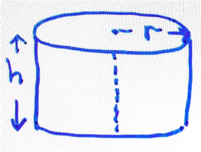

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4. $A = 2\pi r^2 + 2\pi \cdot \dfrac{100}{\pi r^2} \cdot r = 2\pi r^2 + 200 \cdot r^{-1}.$

5. $\frac{dA}{dr} = 4\pi r - 200 \cdot r^{-2} = 0 \Leftrightarrow 4\pi r = \frac{200}{r^2} \Leftrightarrow \pi r^3 = 50$
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The total amount of metal is minimized by a height of $2\sqrt[3]{50/\pi} \approx 5.030$ cm and a radius of $\sqrt[3]{50/\pi} \approx 2.515$ cm.

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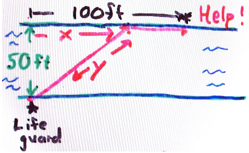

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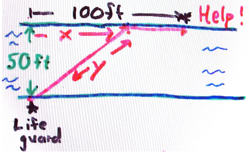

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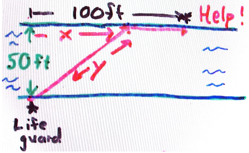

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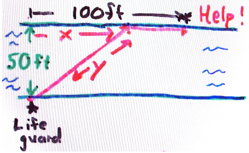

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4.
$T = \frac{1}{5}\sqrt{x^2 + 50^2} + \frac{1}{15}(100 - x)$

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7. Life guard should swim straight to the point $x = \frac{25}{\sqrt{2}} \approx 17.68$ and then run to the person.

Mathematics Saves Life!

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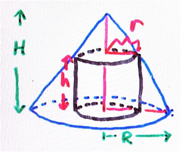

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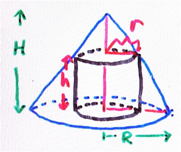

2. $H =$ height of cone, $R =$Radius of cone (given constants)
$h =$ height of cylinder, $r =$radius of cylinder (asked for)
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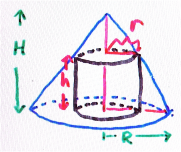

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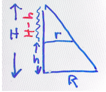

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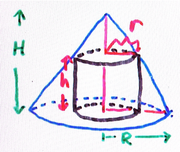

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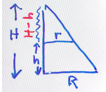

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$$V = \pi \frac{R^2}{H^2}(H-h)^2 h$$

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6. Since $V = 0$ for $h = 0$ or $h = H$ and $V > 0$ for h between, $h = \frac{1}{3}H$ is the absolute maximum.

7. $r = \frac{R}{H}(H - \frac{1}{3}H) = \frac{2}{3}R$.

5. $\dfrac{dV}{dh} = \pi \dfrac{R^2}{H^2}\left[2(H-h)(-1)h + (H-h)^2\right] = 0$

$\Leftrightarrow (H-h)\,[-2h + (H-h)] = (H-h)\,[H-3h] = 0$

$\Leftrightarrow h = H$ or $H - 3h = 0$, $h = \frac{1}{3}H$.

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The volume of the cylinder is maximized for the height $h = \frac{1}{3}H$ and radius $r = \frac{2}{3}R$.