# Calculus I - Lecture 18 - Curve Sketching

Lecture Notes:
**http://www.math.ksu.edu/~gerald/math220d/**

Course Syllabus:
**http://www.math.ksu.edu/math220/spring-2014/indexs14.html**

Gerald Hoehn (based on notes by T. Cochran)

March 31, 2014

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**First derivative test:**

Let $f'(c) = 0$. Then:

$f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c \Rightarrow f(c)$ is local max.
$f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c \Rightarrow f(c)$ is local min.

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**Second derivative test:**

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Those are the points where the graph of $f(x)$ may changes its features. We will concentrate to find those points

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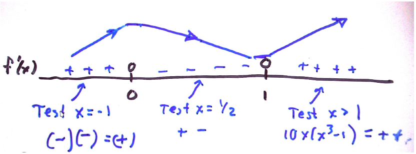

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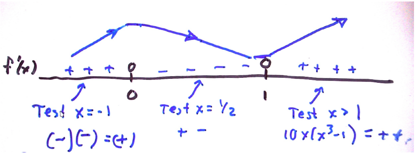


3. $f''(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}(10x^4 - 10x) = 4 \cdot 10x^3 - 10 = 10(4x^3 - 1)$

$f''(x)$ when $4x^3 = 1 \Leftrightarrow x^3 = 1/4 \Leftrightarrow x = 1/\sqrt[3]{4}$.

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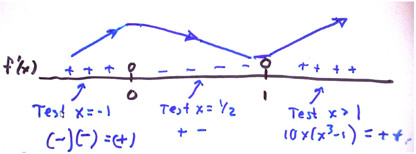


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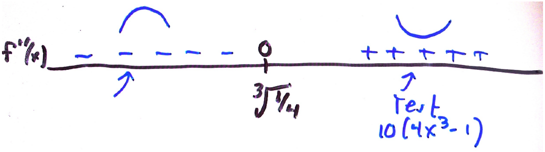

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| $x$ | $f(x) = 2x^5 - 5x^2 + 1$ |
|---|---|
| $0$ | $1$ |
| $1$ | $2 - 5 + 1 = -2$ |
| $1/\sqrt[3]{4}$ | $-.78$ |

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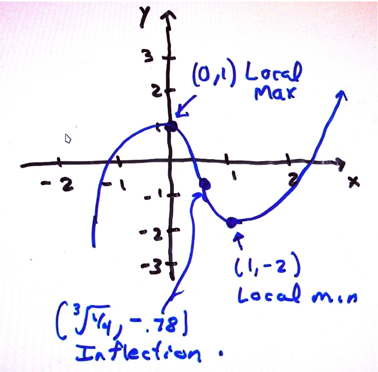

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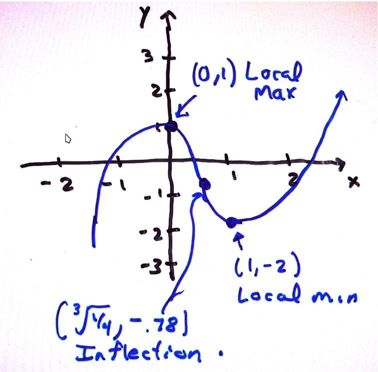


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$$= \frac{3(x-2)}{3(x-2)^{1/3}} + \frac{2x}{3(x-2)^{1/3}} = \frac{5x-6}{3(x-2)^{1/3}}$$

$f'(x) = 0 \Leftrightarrow 5x = 6 \Leftrightarrow x = 6/5$

$f'(x)$ D.N.E. at $x = 2$.

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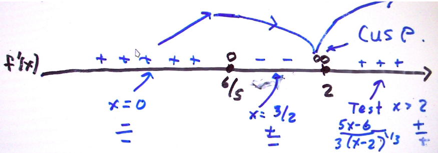

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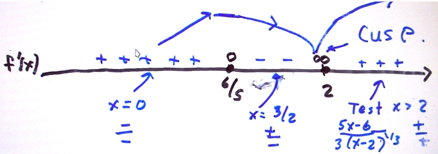


3. $f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{5x-6}{3(x-2)^{1/3}}\right)$

$$= \frac{5 \cdot 3(x-2)^{1/3} - (5x-6)(x-2)^{-2/3}}{9(x-2)^{2/3}}$$

$$= \frac{15(x-2) - (5x-6)}{9(x-2)^{4/3}} = \frac{10x-24}{9(x-2)^{4/3}}$$

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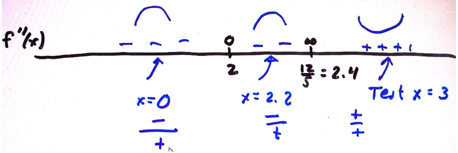

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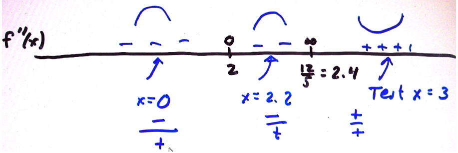


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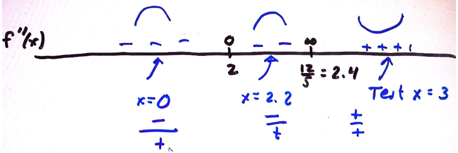


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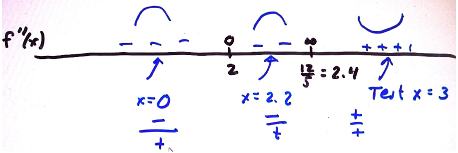


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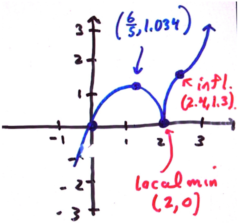

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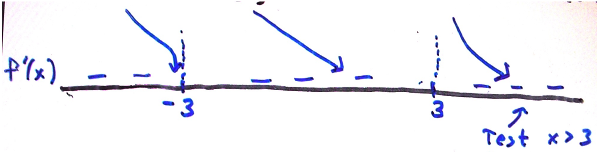

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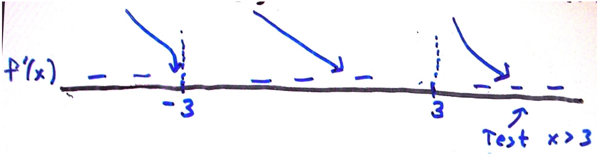


3. $f''(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(-\dfrac{x^2 + 9}{(x^2 - 9)^2}\right)$

$$= \frac{-2x(x^2 - 9)^2 + (x^2 + 9)2(x^2 - 9)2x}{(x^2 - 9)^4}$$

$$= \frac{-2x(x^2 - 9) + (x^2 + 9)2 \cdot 2x}{(x^2 - 9)^3} = \frac{2x(x^2 + 27)}{(x^2 - 9)^3}$$

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Horizontal asymptote $y = 0$.

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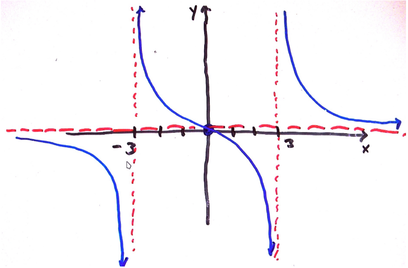

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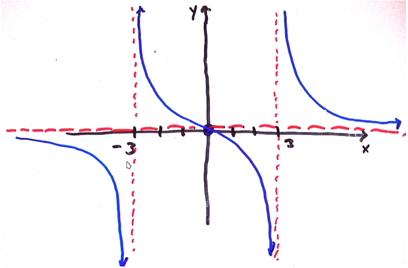


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