Calculus I - Lecture 14 - Related Rates

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

Gerald Hoehn (based on notes by T. Cochran)

March 10, 2014

Problem 8 of Exam 2

Find the derivative, simplify, and determine where it is zero.
$y = \ln(3 + e^{\cos(5x)})$.

Solution with Mathematica

```
Mathematica 7.0 for Sun Solaris SPARC (64-bit)
Copyright 1988-2008 Wolfram Research, Inc.

In[1]:= y=Log[3+E^Cos[5x]]

                   Cos[5 x]
Out[1]= Log[3 + E         ]

In[2]:= D[y,x]

             Cos[5 x]
        -5 E          Sin[5 x]
Out[2]= ---------------------
                   Cos[5 x]
             3 + E

In[3]:= Reduce[%==0,x]

                                               2 Pi C[1]          Pi + 2 Pi C[1]
Out[3]= C[1] \[Element] Integers && (x == --------- || x == --------------)
                                                   5                    5
```

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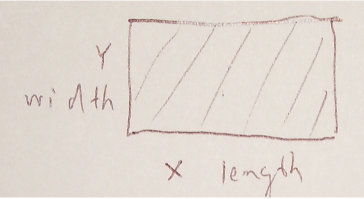

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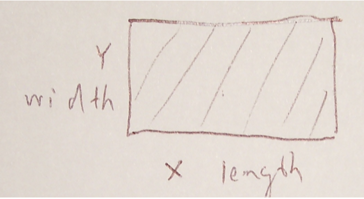

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Given: $\frac{\mathrm{d}x}{\mathrm{d}t} = 5$ ft/sec $\frac{\mathrm{d}y}{\mathrm{d}t} = -2$ ft/sec

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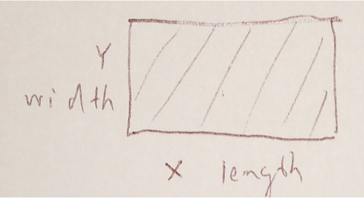

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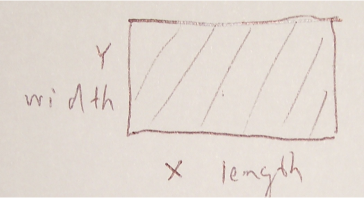

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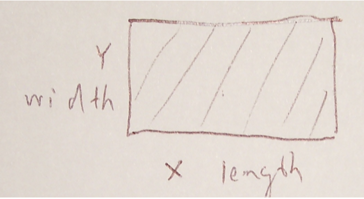

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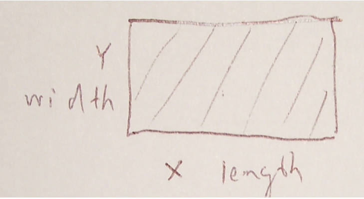

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$\dfrac{dA}{dt} = 5 \cdot 8 + 10 \cdot (-2) = 40 - 20 = 20.$

The area is increasing by a rate of 20 $\mathrm{ft}^2/\mathrm{sec}$.

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7. State the answer in a complete sentence with the correct units.

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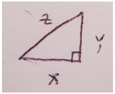
1) Pythagorean Theorem 

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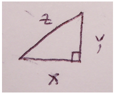
1) Pythagorean Theorem 

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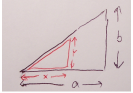
2) Similar Triangles 

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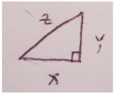
1) Pythagorean Theorem 

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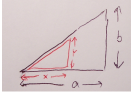
2) Similar Triangles 

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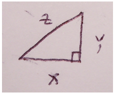
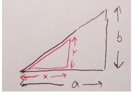
1) Pythagorean Theorem  $x^2 + y^2 = z^2$

2) Similar Triangles  $\frac{y}{b} = \frac{x}{a}$ or $\frac{y}{x} = \frac{b}{a}, \ldots$

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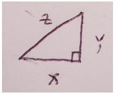
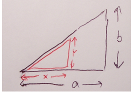
1) Pythagorean Theorem  $x^2 + y^2 = z^2$

2) Similar Triangles  $\frac{y}{b} = \frac{x}{a}$ or $\frac{y}{x} = \frac{b}{a}, \ldots$

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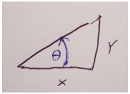
4) Trigonometric Relations  $\frac{y}{x} = \tan\theta$

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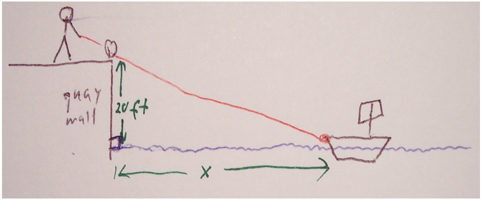
1) 

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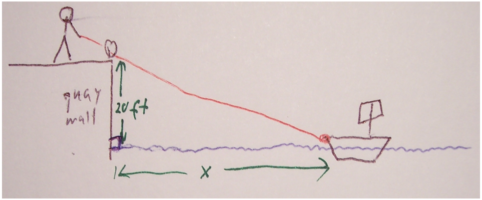
1) 

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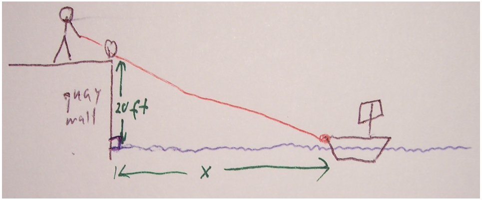
1) 

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3) $z^2 = x^2 + 20^2$ (Pythagorean Theorem)

4)

$$\frac{\mathrm{d}}{\mathrm{d}t} z^2 = \frac{\mathrm{d}}{\mathrm{d}t}(x^2 + 20^2)$$

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5) We need also z.

If $x = 100$ then $z^2 = 100^2 + 20^2 = 10400$.

Thus $z = \sqrt{10400} = 10\sqrt{104} = 20\sqrt{26}$

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7) The boat approaches the shore by the rate $3\sqrt{26}$ ft/min.

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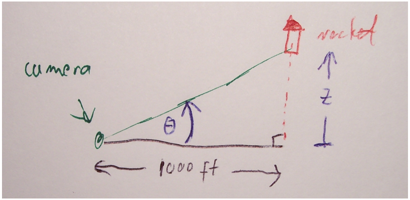
1) 

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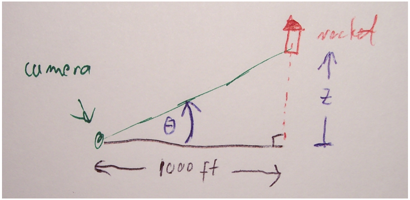
1) 

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Find: $\dfrac{dz}{dt}$ when $\theta = 45^\circ = \frac{\pi}{4}$.

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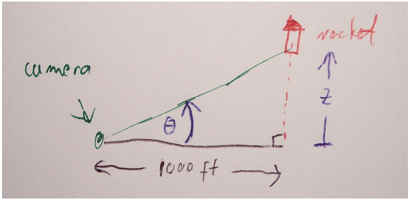
1) 

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3) $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{z}{1000}$

4) $$\frac{\mathrm{d}}{\mathrm{d}t}\tan\theta = \frac{\mathrm{d}}{\mathrm{d}t}\frac{z}{1000}$$

$$\sec^2\theta \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{1000}\frac{\mathrm{d}z}{\mathrm{d}t}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = 1000 \cdot \sec^2\theta \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} = 1000 \cdot \frac{1}{\cos^2\theta} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

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6) $$\left.\frac{\mathrm{d}z}{\mathrm{d}t}\right|_{\theta=\frac{\pi}{4}} = 1000 \cdot \frac{1}{\cos^2(\pi/4)} \cdot \frac{\pi}{9}$$

$$= 1000 \cdot \frac{1}{(1/\sqrt{2})^2} \cdot \frac{\pi}{9} = \frac{2000\pi}{9}$$

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7) The rocket is ascending with a velocity of $\dfrac{2000\pi}{9}$ ft/sec.

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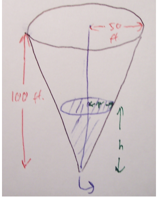
1) 

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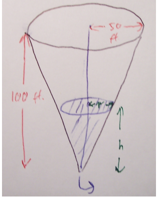
1) 

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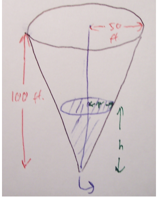
1) 

V Volume of water

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Find: $\frac{\mathrm{d}V}{\mathrm{d}t}$ when $h = 10$ ft.

3) $V = \frac{1}{3}\pi r^2 h$ (Volume of cone)

$\frac{r}{h} = \frac{50}{100} = \frac{1}{2}$ (similar triangles)

4) $$\frac{\mathrm{d}}{\mathrm{d}t}V = \frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{3}\pi r^2 h)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}V = \frac{1}{3}\pi\left(2r \cdot \frac{\mathrm{d}r}{\mathrm{d}t} \cdot h + r^2 \cdot \frac{\mathrm{d}h}{\mathrm{d}t}\right)$$

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5) Need r and $\frac{\mathrm{d}r}{\mathrm{d}t}$.

$\frac{r}{h} = \frac{50}{100} = \frac{1}{2}$ gives $r = \frac{1}{2}h = \frac{10}{2} = 5$ and

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7) The water is flowing out of the tank with $50\pi\ \mathrm{ft}^3/\mathrm{hr}$.

Example: Recall that in baseball the home plate and the three bases form a square of side length 90 ft. A batter hits the ball and runs to the first base at 24 ft/sec. At what rate is his distance from the 2nd base decreasing when he is halfway to the first base.

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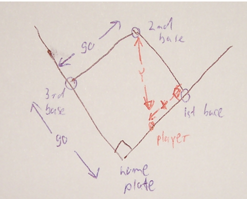
1) 

x distance between player and first base.

y distance between player and 2nd base.

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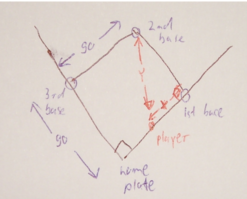
1) 

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2) Given: $\frac{\mathrm{d}x}{\mathrm{d}t} = -24$ ft/sec.

Find: $\frac{\mathrm{d}y}{\mathrm{d}t}$ when $x = \frac{90}{2} = 45$ ft.

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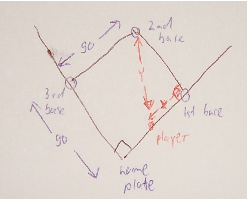
1) 

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2) Given: $\frac{dx}{dt} = -24$ ft/sec.

Find: $\frac{dy}{dt}$ when $x = \frac{90}{2} = 45$ ft.

3) $$y^2 = x^2 + 90^2$$

4) $$\frac{\mathrm{d}}{\mathrm{d}t}(y^2) = \frac{\mathrm{d}}{\mathrm{d}t}(x^2 + 90^2)$$

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$$x = 45, \qquad y = \sqrt{x^2 + 90^2} = \sqrt{45^2 + 90^2} = 45\sqrt{5}$$

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7) The distance between the player and the 2nd base decreases by a rate of $\frac{24}{\sqrt{5}}$ ft/sec.

Example: Grain flows into a conical pile such that the height increases 2 ft/min while the radius increases 3 ft/min. At what rate is the volume increasing when the pile is 2 feet high and has a radius of 4 feet.

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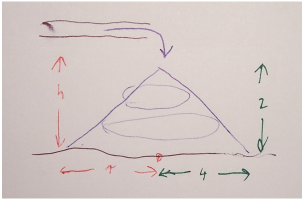
1) 

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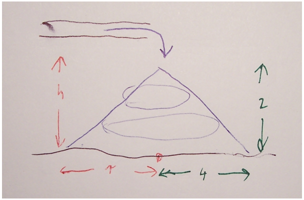
1) 

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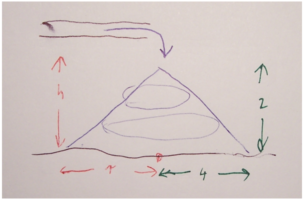
1) 

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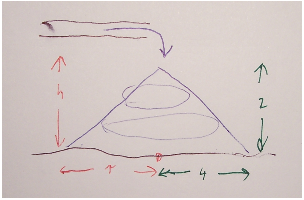
1) 

2) $\frac{dh}{dt} = 2$ ft/min when $h = 2$ ft

$\frac{dr}{dt} = 3$ ft/min when $r = 4$ ft

3) Volume of the cone $V = \frac{1}{3}\pi r^2 h$

4) $$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{3}\pi r^2 h\right) = \frac{1}{3}\pi\left[2r \cdot \frac{\mathrm{d}r}{\mathrm{d}t} \cdot h + r^2 \cdot \frac{\mathrm{d}h}{\mathrm{d}t}\right]$$

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7) The volume is increasing at a rate of $\frac{80\pi}{3}$ $\mathrm{ft}^3/\mathrm{min}$.