Calculus I - Lecture 13 - Review Exam 2

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

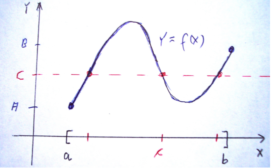
Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

Gerald Hoehn (based on notes by T. Cochran)

March 5, 2014

Theorem (Intermediate Value Theorem (IVT))

Let $f(x)$ be **continuous** *on the interval $[a, b]$ with $f(a) = A$ and $f(b) = B$.*



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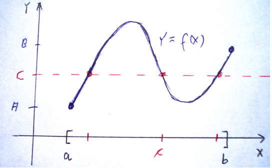

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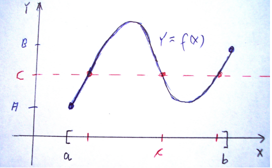

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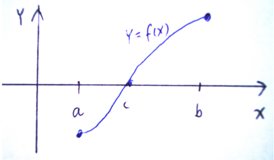
Important special case of the IVT:

Suppose that $f(x)$ is **continuous** on the interval $[a, b]$ with $f(a) < 0$ and $f(b) > 0$.

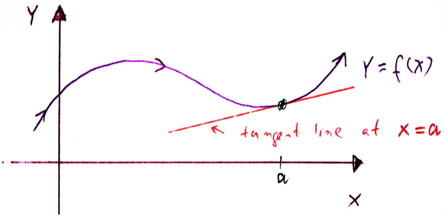
Then there is a point $c \in [a, b]$ where $f(c) = 0$.



Geometric View of the Derivative

Recall, the slope of a line is

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x}$$



Definition (Tangent Line)

A tangent line is a line that (in general)

1. *touches the graph at one point (near that point) and*
2. *has a slope equal to the slope of the curve.*

If the curve is a line segment, the tangent line coincides with the segment.

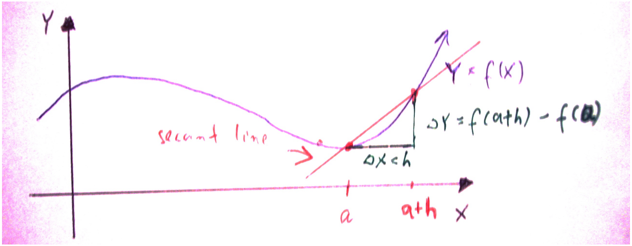
Slope of a curve at $x = a$ equals m_{tan} = slope of tangent line.

Definition (Derivative — geometric)

The **derivative** *of a function $f(x)$ at $x = a$, denoted $f'(a)$ (pronounced "f prime of a"), is the slope of the curve $y = f(x)$ at $x = a$.*

$$\begin{aligned} f'(a) &= \text{the derivative of f(x) at } a \\ &= m_{\tan}, \text{ the slope of the tangent line.} \end{aligned}$$

Algebraic View of the Derivative



Let us determine the slope of the curve at $x = a$.

$$
\begin{aligned}
\text{Let } h &= \text{tiny positive number (e.g. 0.0001)} \\
m_{\text{sec}} &= \text{slope of the secant line shown above} \\
&= \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h} \\
m_{\text{tan}} &= \lim_{h \to 0} m_{\text{sec}}
\end{aligned}
$$

Definition (Derivative — algebraic)

$$
f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
$$

Important Derivatives

$f(x)$	$f'(x)$	
c	0	(c any real constant)
x	1	
x^n	$n\,x^{n-1}$	(n any real constant)
e^x	e^x	
b^x	$(\ln b)\, b^x$	(b any positive constant)
$\ln x$	$\frac{1}{x}$	
$\log_b x$	$\frac{1}{\ln b}\,\frac{1}{x}$	(b any positive constant)
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\tan x$	$\sec^2 x$	
$\sec x$	$\sec x\,\tan x$	
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	
$\arctan x$	$\frac{1}{1+x^2}$	

Important Rules

Sum and Difference Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x) \pm g(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) \pm \frac{\mathrm{d}}{\mathrm{d}x} g(x)$$

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Inverse Function Rule

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

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Since $f(x)$ is a continuous function, it has by the intermediate value theorem a zero on the interval $[-2, 2]$.