Calculus I - Lecture 12 - Implicit Differentiation

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

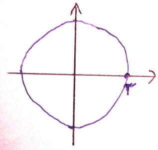
Gerald Hoehn (based on notes by T. Cochran)

March 3, 2014

Implicit Differentiation

Implicit differentiation is a method for finding the slope of a curve, when the equation of the curve is not given in “explicit” form $y = f(x)$, but in “implicit” form by an equation $g(x, y) = 0$.

Examples



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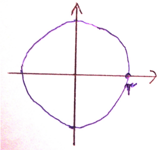
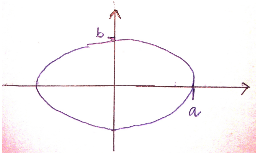


2) Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



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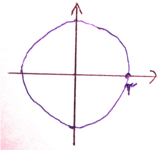

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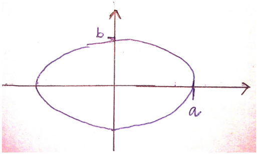
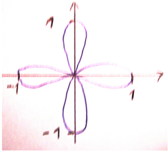


3) 4-leaf clover $(x^2 + y^2)^3 = (x^2 - y^2)^2$



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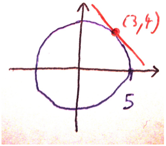
$$2y \cdot y' = -2x$$

$$y' = -\frac{2x}{2y} = -\frac{x}{y}$$

Example:

b) What is the slope of the circle at $(3, 4)$?



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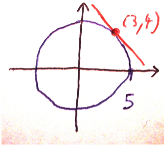

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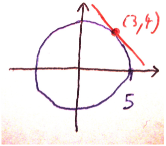

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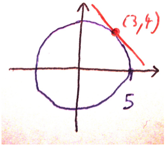

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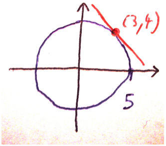

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The curve has a vertical tangent at $(5, 0)$.

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Exponential Rule

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{g(y)} = e^{g(y)} \cdot g'(y) \cdot y'$$

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Make sure that there is no y' left on right-hand side.

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$$y'\Big|_{(1,2)} = \frac{1 - 2 \cdot 1 \cdot 2}{1^2 - 3 \cdot 2^2} = \frac{-3}{-11} = \frac{3}{11}$$

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Point slope form of tangent line: $y = m(x - x_1) + y_1$

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Point slope form of tangent line: $y = m(x - x_1) + y_1$

$$y = \tfrac{3}{11}(x - 1) + 2 = \tfrac{3}{11}x + \tfrac{19}{11}$$

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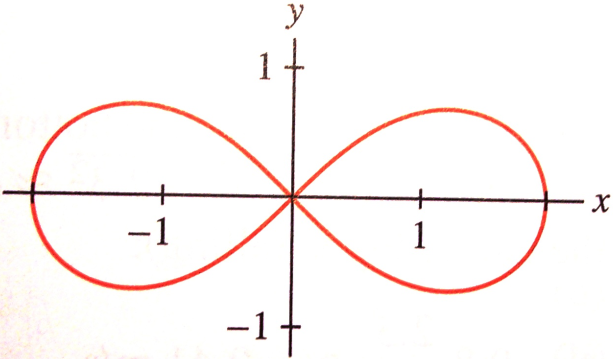
Only the four points $(\pm\sqrt{3/2}, \pm\sqrt{1/2})$ but not $(0, 0)$ are solutions.



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