Calculus I - Lecture 11 - Derivatives of General Exponential and Inverse Functions

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

Gerald Hoehn (based on notes by T. Cochran)

February 26, 2014

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Theorem

$$\frac{\mathrm{d}}{\mathrm{d}x} b^x = (\ln b) \cdot b^x, \quad \textit{for any base } b > 0.$$

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We have shown the following rule:

Theorem

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Example: Compute $\frac{\mathrm{d}}{\mathrm{d}x} \log_5(\log_5(x))$.

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x} \log_5(\log_5(x)) = \frac{1}{\ln 5} \cdot \frac{1}{\log_5 x} \cdot \frac{\mathrm{d}}{\mathrm{d}x} \log_5 x$$

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Let $y = f(x)$, $y' = \frac{dy}{dx} = f'(x)$.

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3. Solve for y'.

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$$\begin{aligned}\ln y &= \ln\left(\sqrt[4]{x}\sin^3 x\right)\\ &= \ln(\sqrt[4]{x}) + \ln(\sin^3 x)\\ &= \ln(x^{1/4}) + \ln((\sin x)^3)\end{aligned}$$

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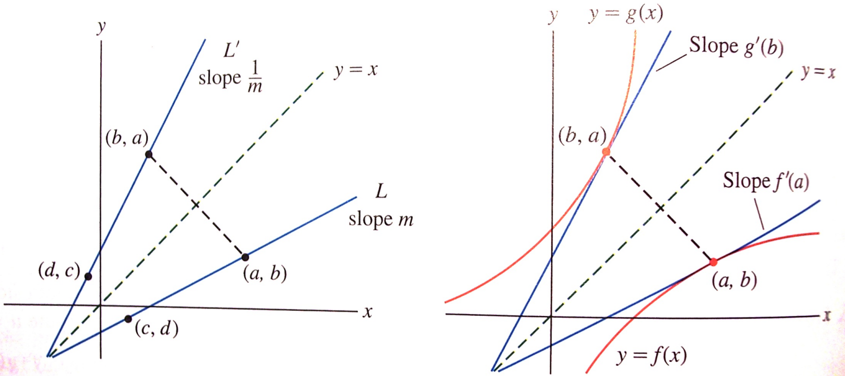
Graphical Understanding



(A) If L has slope m, then its reflection L' has slope $1/m$.

(B) The tangent line to the inverse $y = g(x)$ is the reflection of the tangent line to $y = f(x)$.

Inverses of the trigonometric functions

$y = \sin^{-1}(x) = \arcsin(x)$, $-1 \leq x \leq 1$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

inverse sine function or arcsine function

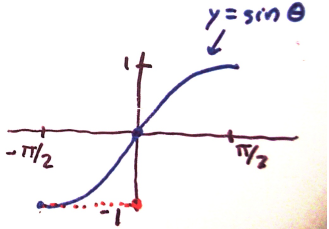
$\sin^{-1}(x)$ = angle (or arc) between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x.

Note: The -1 in the exponent means inverse function, not $\frac{1}{\sin x}$.

ex. $\sin^{-1}(0) = 0$

$\sin^{-1}(-1) = -\pi/2$

$\sin^{-1}(1/\sqrt{2}) = \pi/4$



Inverses of the trigonometric functions

Inverse cosine:

$y = \cos^{-1}(x) = \arccos(x)$, $-1 \le x \le 1$, $0 \le y \le \pi$

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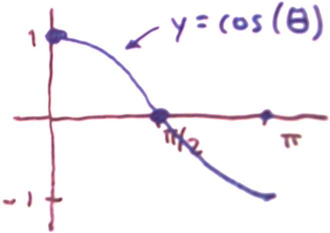

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$f(x)$	$f'(x)$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos x$	$\dfrac{-1}{\sqrt{1-x^2}}$
$\arctan x$	$\dfrac{1}{1+x^2}$
$\mathrm{arcsec} x$	$\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$