Calculus I - Lecture 8 - The derivative function A

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

Gerald Hoehn (based on notes by T. Cochran)

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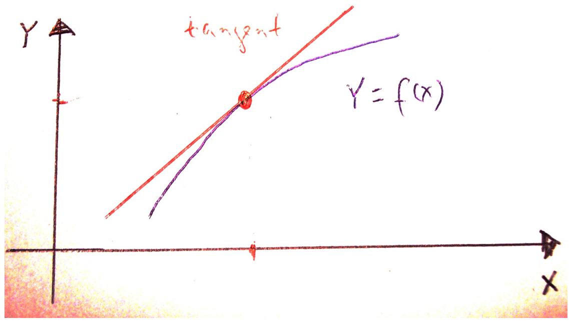
Section 3.2 – Derivative as a function

Last time we saw the geometric and algebraic definition of the derivative of a function $f(x)$ at a point $x = a$



$$f'(a) = \text{slope of tangent line} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

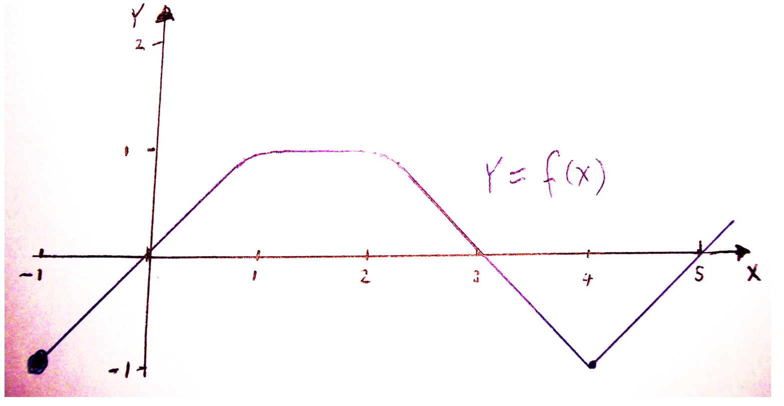
Definition

The **derivative function** $f'(x)$ *of a function* $f(x)$ *is the function which has the value* $f'(a)$ *for* $x = a$*.*

The **domain** of $f'(x)$ consists of all values of x in the domain of $f(x)$ for which the limit defining $f'(a)$ exists.

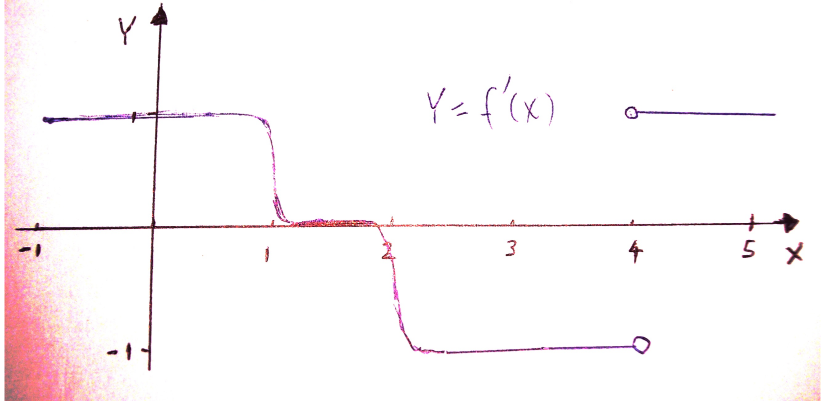
We say $f(x)$ is **differentiable** on (a, b) if it is defined there. If $f'(x)$ exists for all x in the domain of $f(x)$ we simply say $f(x)$ is differentiable.

Example: (Graphical determination of the derivative)
Plot the graph of $f'(x)$ for the function $f(x)$ given by the following graph:



Solution:

x	$-.5$	0	$.5$	1.5	2.5	3	4	5
$f'(x)$	1	1	1	0	−1	−1	D.N.E.	1

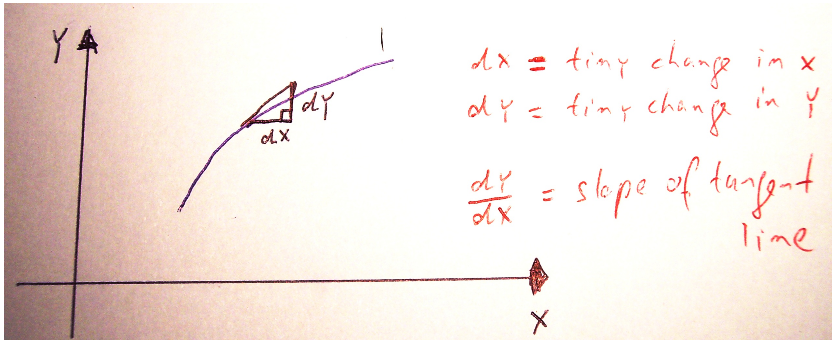


Today we will start building up a collection of rules ("short-cuts") for calculating derivatives.

Leibniz Notation: If $y = f(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} f(x) = f'(x) = y'.$$

$\frac{\mathrm{d}y}{\mathrm{d}x}$ reminds us about the slope of the tangent:



$\frac{\mathrm{d}}{\mathrm{d}x}$ has the operator symbol meaning:
"take the derivative of the function $f(x)$"

Warmup

Example: a) Find the derivative of $f(x) = x^2$ using the limit definition, and display the answer in Leibniz notation.

Solution:

$$f'(x) = \lim_{h\to0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to0} \frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to0} \frac{h(2x+h)}{h} = 2x$$

$$\frac{\mathrm{d}}{\mathrm{d}x} x^2 = 2x \qquad \text{(Leibniz Notation)}$$

b) Find $\frac{\mathrm{d}}{\mathrm{d}x} x^2\big|_{x=3}$. (" $\ldots\big|_{x=a}$ " = evaluate "$\ldots$ at $x = a$")

Solution: $\frac{\mathrm{d}}{\mathrm{d}x} x^2\big|_{x=3} = 2x\big|_{x=3} = 2\cdot3 = 6$

c) Find the slope of the curve $y = x^2$ at $x = -2$.

Solution: $\frac{\mathrm{d}}{\mathrm{d}x} x^2\big|_{x=-2} = 2x\big|_{x=-2} = 2\cdot(-2) = -4$

Rules for Derivatives

Rule 1: $\frac{\mathrm{d}}{\mathrm{d}x} c = 0$ (c a constant)

Example: a) $\frac{\mathrm{d}}{\mathrm{d}x} 17 = 0$ b) $\frac{\mathrm{d}}{\mathrm{d}x} \sqrt{2} = 0$

Rule 2: Power Rule. For every real number *n*

$$\frac{\mathrm{d}}{\mathrm{d}x} x^n = n\,x^{n-1} \quad \text{(where } x \text{ is defined)}$$

Example: a) $\frac{\mathrm{d}}{\mathrm{d}x} x^7 = 7\,x^{7-1} = 7\,x^6$

b) $\frac{\mathrm{d}}{\mathrm{d}x} x = \frac{\mathrm{d}}{\mathrm{d}x} x^1 = 1\,x^0 = 1$

c) $\frac{\mathrm{d}}{\mathrm{d}x} \sqrt[3]{x} = \frac{\mathrm{d}}{\mathrm{d}x} x^{1/3} = \frac{1}{3} x^{1/3-1} = \frac{1}{3} x^{-2/3}$

d) $\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{x^5} = \frac{\mathrm{d}}{\mathrm{d}x} x^{-5} = -5\,x^{-5-1} = -5\,x^{-6}$

Rule 3: Sum and Difference Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x) \pm g(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) \pm \frac{\mathrm{d}}{\mathrm{d}x} g(x)$$

Example: $\frac{\mathrm{d}}{\mathrm{d}x}(x^3 - x^5) = \frac{\mathrm{d}}{\mathrm{d}x} x^3 - \frac{\mathrm{d}}{\mathrm{d}x} x^5 = 3\,x^2 - 5\,x^4$

Rule 4: Constant Factor Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(c \cdot f(x)\right) = c\,\frac{\mathrm{d}}{\mathrm{d}x} f(x) \qquad (c \text{ a constant})$$

Example:

a) $$\frac{\mathrm{d}}{\mathrm{d}x}\left(7\,x^3 - \frac{1}{x} + 5\right) = \frac{\mathrm{d}}{\mathrm{d}x} 7\,x^3 - \frac{\mathrm{d}}{\mathrm{d}x} x^{-1} + \frac{\mathrm{d}}{\mathrm{d}x} 5 \quad \text{(Rule 3)}$$

$$= 7\,\frac{\mathrm{d}}{\mathrm{d}x} x^3 - \frac{\mathrm{d}}{\mathrm{d}x} x^{-1} + \frac{\mathrm{d}}{\mathrm{d}x} 5 \quad \text{(Rule 4)}$$

$$= 7 \cdot 3\,x^2 - (-1)\,x^{-2} + 0$$

$$= 21\,x^2 + x^{-2}$$

b) $$\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\pi t} = \sqrt{\pi}\,\frac{\mathrm{d}}{\mathrm{d}t} t^{1/2} = \sqrt{\pi}\,\frac{1}{2} t^{-1/2} = \frac{\sqrt{\pi}}{2\sqrt{t}}$$

Example: Find $\frac{\mathrm{d}}{\mathrm{d}x}\left((x - x^3)(\frac{1}{x} + x^4)\right)$.

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((x - x^3)(\frac{1}{x} + x^4)\right) = \frac{\mathrm{d}}{\mathrm{d}x}(1 - x^2 + x^5 - x^7)$$
$$= -2x + 5\,x^4 - 7x^6$$

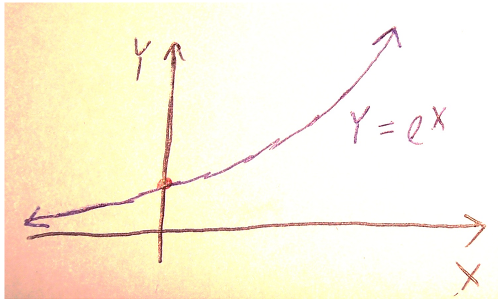
Rule 5: Exponential Function

$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x \qquad (\mathrm{e}=2.718281...)$$

The derivative of the exponential function is itself.

Example: Find the slope of the curve $y = e^x$ at $x = 0$.

Solution:



$$\frac{\mathrm{d}}{\mathrm{d}x}e^x\Big|_{x=0} = e^x\Big|_{x=0} = e^0 = 1$$

Section 3.3 – Product and Quotient Rule

Rule 6: Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\cdot g(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x}f(x)\cdot g(x) + f(x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}g(x)$$

Warning: $(f(x)g(x))' \neq f'(x)\cdot g'(x)$

Example:

a) $\frac{\mathrm{d}}{\mathrm{d}x}(x^7e^x) = \frac{\mathrm{d}}{\mathrm{d}x}x^7\cdot e^x + x^7\cdot\frac{\mathrm{d}}{\mathrm{d}x}e^x$

$$= 7x^6e^x + x^7e^x$$

$$= (7x^6 + x^7)e^x$$

b) $\frac{\mathrm{d}}{\mathrm{d}x}\left((x - x^3)(\frac{1}{x} + x^4)\right)$

$$= \frac{\mathrm{d}}{\mathrm{d}x}(x - x^3)\cdot(\frac{1}{x} + x^4) + (x - x^3)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{x} + x^4)$$

$$= (1 - 3x^2)(x^{-1} + x^4) + (x - x^3)(-x^{-2} + 4x^3)$$

$$= (x^{-1} - 3x + x^4 - 3x^6) + (-x^{-1} + x + 4x^4 - 4x^6)$$

$$= -2x + 5x^4 - 7x^6$$

Rule 7: Quotient Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{\mathrm{d}}{\mathrm{d}x}f(x)\cdot g(x) - f(x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}g(x)}{g(x)^2}$$

Example:

a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2-1}{2x+5}\right)$

$$= \frac{\frac{\mathrm{d}}{\mathrm{d}x}(x^2-1)\cdot(2x+5) - (x^2-1)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(2x+5)}{(2x+5)^2}$$

$$= \frac{2x(2x+5) - (x^2-1)\cdot 2}{(2x+5)^2}$$

$$= \frac{2x^2+10x+2}{(2x+5)^2}$$

b) $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^2e^t}{1+t^2}\right)$

$$= \frac{\frac{\mathrm{d}}{\mathrm{d}t}(t^2e^t)\cdot(1+t^2) - t^2e^t\cdot\frac{\mathrm{d}}{\mathrm{d}t}(1+t^2)}{(1+t^2)^2} \qquad \text{(quotient first)}$$

$$= \frac{(2te^t + t^2e^t)\cdot(1+t^2) - t^2e^t\cdot 2t}{(1+t^2)^2}$$

$$= \frac{(2t + t^2 + t^4)e^t}{(1+t^2)^2}$$