Calculus I - Lecture 7 - The Derivative

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

Gerald Hoehn (based on notes by T. Cochran)

February 12, 2014

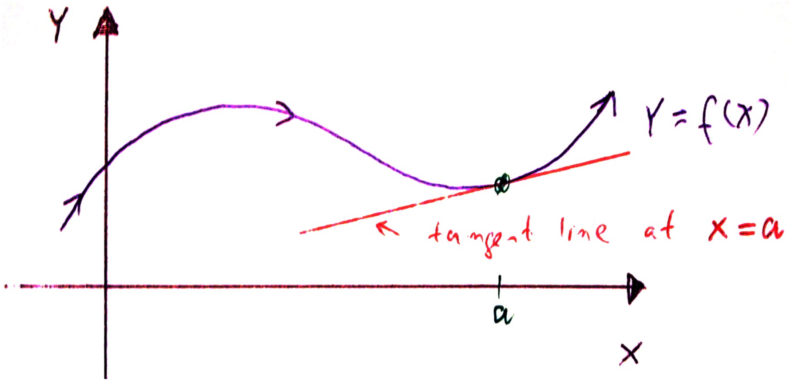
Section 3.1 – Definition of the Derivative

In this section we will give both a **geometric** and an **algebraic** definition of the derivative

Geometric View of the Derivative

Recall, the slope of a line is

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x}$$



Definition (Tangent Line)

A tangent line is a line that (in general)

1. *touches the graph at one point (near that point) and*
2. *has a slope equal to the slope of the curve.*

If the curve is a line segment, the tangent line coincides with the segment.

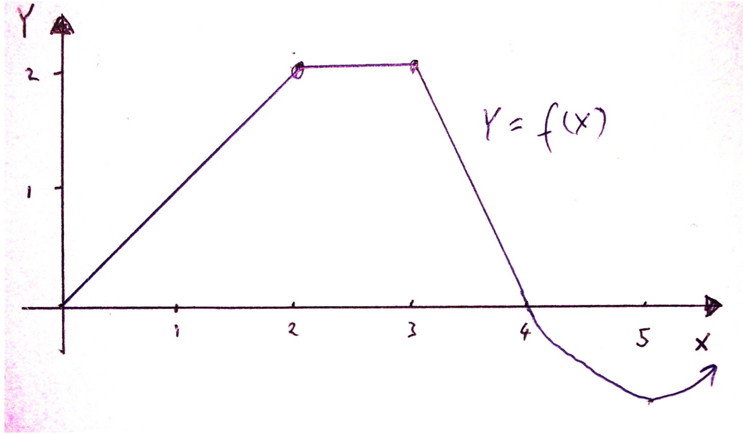
Slope of a curve at $x = a$ equals $m_{\tan} =$ slope of tangent line.

Definition (Derivative — geometric)

The **derivative** *of a function $f(x)$ at $x = a$, denoted $f'(a)$ (pronounced "f prime of a"), is the slope of the curve $y = f(x)$ at $x = a$.*

$$\begin{aligned} f'(a) &= \text{the derivative of f(x) at } a \\ &= m_{\tan}, \text{ the slope of the tangent line.} \end{aligned}$$

Example: Determine by inspection the following derivatives.



a) $f'(1) = 1 \quad (m = 1)$

b) $f'(2.2) = 0 \quad (m = 0)$

c) $f'(\pi) = -2 \quad (m = \frac{\Delta y}{\Delta x} = \frac{-2}{1})$

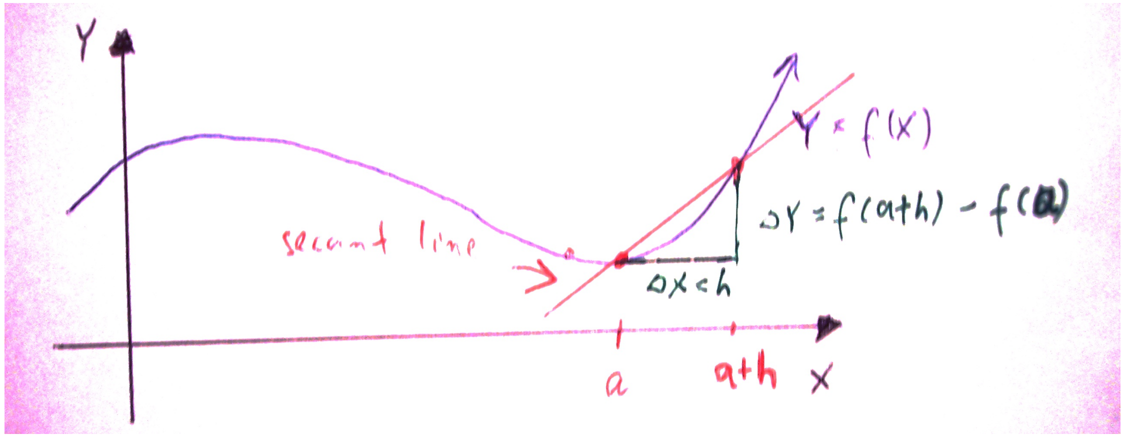
d) $f'(5) = 0 \quad$ (tangent $m = 0$)

e) What about $f'(2)$ and $f'(3)$?

Undefined. Do not exist.

At sharp corners, $f'(a)$ does not exist.

Algebraic View of the Derivative



Let us determine the slope of the curve at $x = a$.

Let $h =$ tiny positive number (e.g. 0.0001)

$m_{\mathrm{sec}} =$ slope of the secant line shown above

$$= \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}$$

$$m_{\mathrm{tan}} = \lim_{h \to 0} m_{\mathrm{sec}}$$

Definition (Derivative — algebraic)

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Memorize this!

Example:

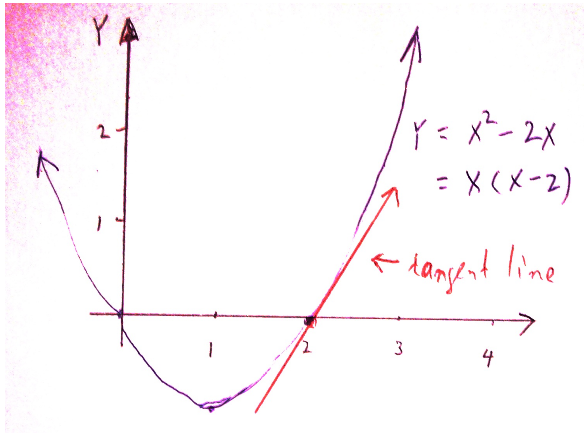
a) Find $f'(2)$ where $f(x) = x^2 - 2x$.

b) Use part a) to find the equation of the tangent line to the curve $y = x^2 - 2x$ at $(2, 0)$.

Solution: a)

$$
\begin{aligned}
f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h \to 0} \frac{[(2+h)^2 - 2(2+h)] - [2^2 - 2 \cdot 2]}{h} \\
&= \lim_{h \to 0} \frac{4 + 4h + h^2 - 4 + h}{h} \\
&= \lim_{h \to 0} \frac{2h + h^2}{h} \\
&= \lim_{h \to 0} (2 + h) = 2
\end{aligned}
$$

Solution: b)



We use point-slope form of the tangent line:

$$y - y_1 = m(x - x_1)$$

$$y - 0 = 2(x - 2) \quad \text{(since } m = 2 \text{ by a))}$$

$$y = 2x - 4$$

Example: Let $f(x) = \sqrt{x}$. Find $f'(a)$, where a is any value > 0.

Solution:

$$
\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h \to 0} \frac{(\sqrt{a+h} - \sqrt{a})}{h} \cdot \frac{(\sqrt{a+h} + \sqrt{a})}{(\sqrt{a+h} + \sqrt{a})} \\
&= \lim_{h \to 0} \frac{(a+h) - \sqrt{a}\sqrt{a+h} + \sqrt{a}\sqrt{a+h} - a}{h(\sqrt{a+h} + \sqrt{a})} \\
&= \lim_{h \to 0} \frac{h}{h(\sqrt{a+h} + \sqrt{a})} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{a+h} + \sqrt{a}} = \frac{1}{\sqrt{a} + \sqrt{a}} = \frac{1}{2\sqrt{a}}
\end{aligned}
$$

If $f(x) = \sqrt{x} = x^{1/2}$, then $f'(a) = \frac{1}{2\sqrt{a}} = \frac{1}{2}a^{-1/2}$.

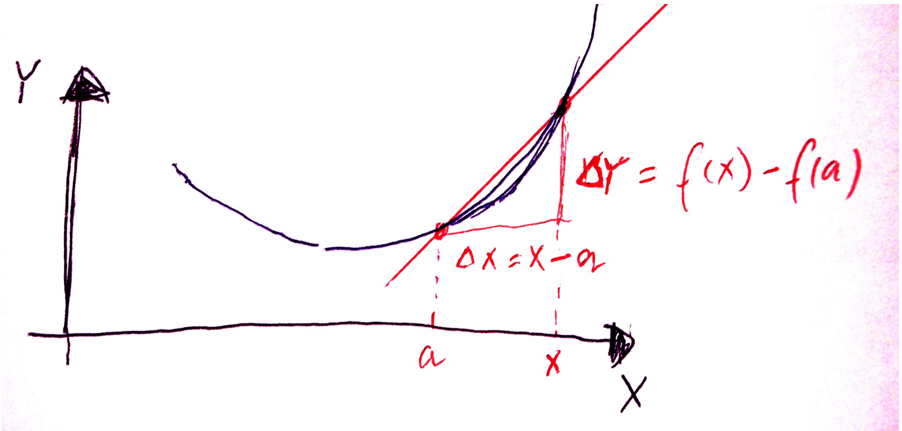
We will see short cuts next time.

Two formulas for $f'(a)$:

1) $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad \text{(as in definition)}$$

2) $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

2) is obtained by letting $h = x - a$, so $x = a + h$, and $h \to 0$ is equivalent to $x \to a$.



$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}.$$

Example: Let $f(x) = \dfrac{1}{x}$. Find $f'(a)$ using method 2.

Solution:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a}$$

$$= \lim_{x \to a} \frac{\frac{a}{a}\frac{1}{x} - \frac{1}{a}\frac{x}{x}}{x - a} = \lim_{x \to a} \frac{\frac{a - x}{ax}}{\frac{x - a}{1}}$$

$$= \lim_{x \to a} \frac{a - x}{ax} \cdot \frac{1}{x - a} = \lim_{x \to a} \frac{-(x - a)}{ax(x - a)}$$

$$= \lim_{x \to a} -\frac{1}{ax} = -\frac{1}{a^2}$$

Note:

We have seen that

1. if $f(x) = \sqrt{x} = x^{1/2}$ then $f'(a) = \frac{1}{2}a^{-1/2}$,
2. if $f(x) = \frac{1}{x} = x^{-1}$ then $f'(a) = (-1) \cdot a^{-2}$.

What is the pattern?

Theorem (Power rule)

Let n be any real number. If $f(x) = x^n$, then $f'(a) = n \cdot a^{n-1}$ for any real number a where $f(x)$ is defined.

Since a is arbitrary, we simply replace a with x (a variable) and say

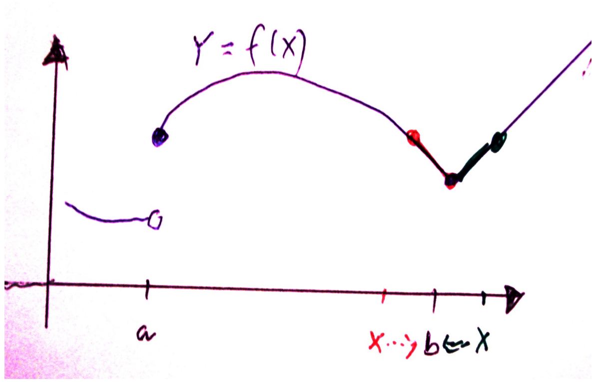
$$f'(x) = n\,x^{n-1}.$$

Note: Intuitively, $f'(a)$ **fails to exist** if either

i) $f(x)$ has a discontinuity at $x = a$, or

ii) the graph of $f(x)$ has a sharp corner at $x = a$.

Example:



$f'(a)$ does not exist since $f(x)$ is not continuous at a.
Try to find $f'(b)$:

$$\lim_{x \to b^+} \frac{f(x) - f(b)}{x - b} = 1$$

$$\lim_{x \to b^-} \frac{f(x) - f(b)}{x - b} = -1$$

The one-sided limits are not equal.
Thus the two-sided limit $f'(b) = \lim_{x \to b} \frac{f(x)-f(b)}{x-b}$ does not exist.