Calculus I - Lecture 5 - Review for Exam 1

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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February 5, 2014

Exam 1

Thursday, Feb. 6 7:05 – 8:20 p.m.

Room assignments:

Room	Last Names
CW 101	A – G
S 063	H – M
WB 123	N – Z

No notes, books, calculators or other electronic devices are permitted on the exam. Bring your **student ID** with you.

For review, do the **Practice Test #1** on the course web-site. Exam and solutions are posted online.

There is also a **SAS review**:
Feb. 5, 6:30 – 8:30 p.m. in 1107 Fiedler.

Facts from Trigonometry

Most important facts from Trigonometry you should know:

1) Definition of $\sin\theta$, $\cos\theta$ (on unit circle)

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}$$

2) Evaluation of trig. functions at 0, $\frac{\pi}{2}$, π, 2π.

3) $\sin^2\theta + \cos^2\theta = 1$

4) $\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ (θ in radians)

Facts from Geometry

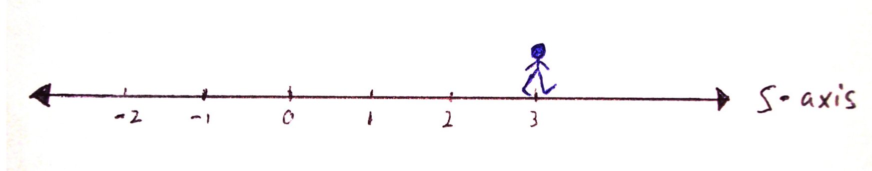
Most important facts from Geometry you should know:

1) Area of square, rectangle, triangle and circle.

2) Circumference of square, rectangle, triangle and circle.

3) Equations for lines and circles and their geometric meaning.

Average and Instantaneous rate of change



s = position of object on the line = $s(t)$: function of time
t = time

$$\text{Average velocity over } [t_1, t_2] \;=\; \frac{s(t_2) - s(t_1)}{t_2 - t_1} \;=\; \frac{\Delta s}{\Delta t}$$

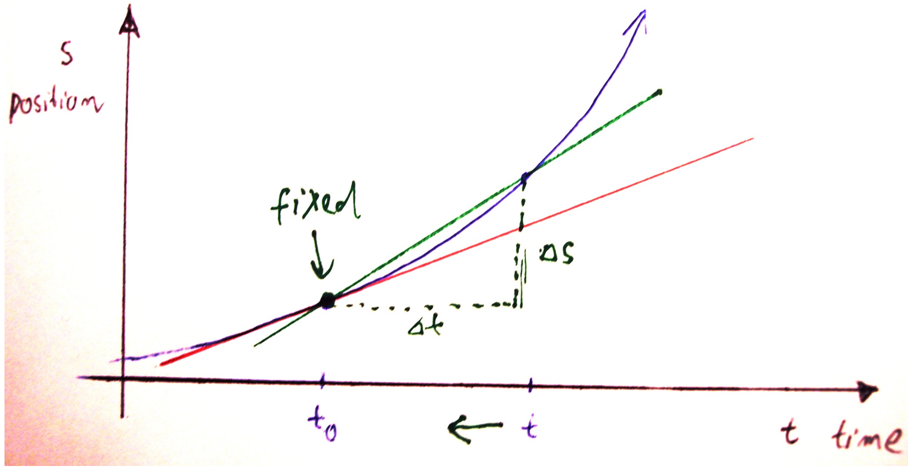
Δs = change in position,
Δs = change in time

Let $v(t_0)$ = instantaneous velocity at t_0,
and v_{ave} = average velocity over the time interval $[t_0, t]$.

$$v(t_0) \;=\; \lim_{t \longrightarrow t_0} v_{\text{ave}} \;=\; \lim_{t \longrightarrow t_0} \frac{s(t) - s(t_0)}{t - t_0}$$

$\lim_{t \longrightarrow t_0} v_{\text{ave}}$ = the value that v_{ave} approaches as t gets closer and closer to t_0.

Graphical interpretation of average and instantaneous velocity



$$v_{\rm ave} = \frac{\Delta s}{\Delta t} = \frac{s(t) - s(t_0)}{t - t_0}$$

$$= \text{slope of “secant” line to graph,}$$

$$v(t_0) = \lim_{t \to t_0} \frac{\Delta s}{\Delta t}$$

$$= \text{slope of the tangent line to the curve at the point } (t_0, s(t_0)).$$

Definition of Limits

$$\lim_{x \to a} f(x) = \text{"limit as } x \text{ approaches } a \text{ of } f(x)\text{"}$$

$:=$ the value that $f(x)$ approaches as x gets closer and closer to a

Notes: 1) x is allowed to approach a from the right or left, **but x is not allowed to equal a.**

2) In order for the limit to exist you must get the **same value** approaching from the left or the right.

One-sided limits:

$\lim_{x \to a^+} f(x) =$ "limit of $f(x)$ as x approaches a from the right"

$\lim_{x \to a^-} f(x) =$ "limit of $f(x)$ as x approaches a from the left"

Limit Laws

Theorem (Basic Limit Laws)

Suppose that $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist. Then:*

(1) $\lim_{x\to a} (f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

(2) $\lim_{x\to a} c\, f(x) = c \cdot \lim_{x\to a} f(x)$

(3) $\lim_{x\to a} (f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

(4) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *(if* $\lim_{x\to a} g(x) \neq 0$*)*

(5) $\lim_{x\to a} f(x)^n = \left(\lim_{x\to a} f(x)\right)^n$

(6) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ *(if* $\lim_{x\to a} f(x) \geq 0$ *when* n *even)*

Continuity

Definition: $f(x)$ is continuous at $x = a$ if

1. $f(a)$ is defined,
2. $\lim_{x \to a} f(x)$ exists, and
3. $\lim_{x \to a} f(x) = f(a)$.

In other words, the limit can be evaluated by plugging in $x = a$.

One sided continuity:

1. $f(x)$ is **continuous from the right** at $x = a$ if
   $$\lim_{x \to a^+} f(x) = f(a)$$
2. $f(x)$ is **continuous from the left** at $x = a$ if
   $$\lim_{x \to a^-} f(x) = f(a)$$

Theorem

The following functions are continuous:

- $f(x) = x^n$ *for integer numbers* n *everywhere on its domain;*
- $f(x) = x^{1/n}$ *for natural numbers* n *everywhere on its domain;*
- $f(x) = b^x$ *for* $b > 0$ *on the whole real line;*
- $f(x) = \log_b x$ *for* $b > 0$ *and* $x > 0$*;*
- $f(x) = \sin x$ *and* $f(x) = \cos x$ *on the whole real line.*

Theorem

Let $f(x)$ *and* $g(x)$ *be continuous at* $x = c$*. Then also the following functions are continuous at* $x = c$*:*

$$f(x) \pm g(x); \qquad f(x) \cdot g(x); \qquad \frac{f(x)}{g(x)} \text{ if } g(c) \neq 0.$$

Theorem

If $g(x)$ *is continuous at* $x = c$ *and* $f(x)$ *is continuous at* $x = g(c)$*, then the composite function* $F(x) = f(g(x))$ *is continuous at* $x = c$*.*

Calculating Limits

There are two main types of limits:

1. **Plug-in Types**

   Generally this is the case if there is no denominator approaching 0.

2. $\frac{0}{0}$ **type Limits**

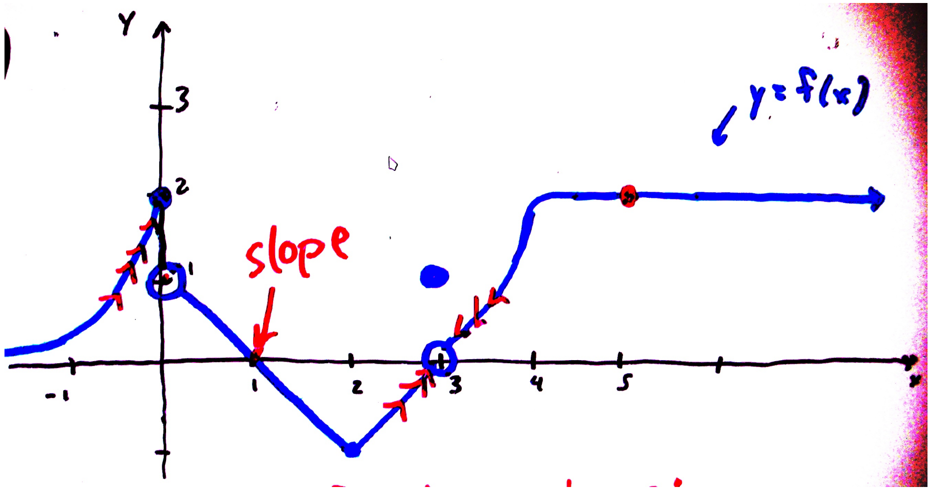
   Numerator and denominator approach 0.

Always start any limit problem with a plug-in test.

Three tricks for $\frac{0}{0}$ type limits

1. Factor top and bottom and cancel the factor approaching 0.
2. Do some algebraic simplification first.
3. Use conjugate trick.

Example:



a) Evaluate $\lim_{x \to 0^-} f(x) \ = \ 2$

b) Evaluate $\lim_{x \to 0} f(x) \ = \ $ D.N.E., left and right limit are different

c) Evaluate $\lim_{x \to 3} f(x) \ = \ 0$

d) Evaluate $\lim_{x \to 5} f(x) \ = \ 2$

e) Where is $f(x)$ **not** continuous? $x = 0, 3$

Evaluate $\lim_{x\to 3} \dfrac{x-3}{x^2-4x+3}$ (show work):

Solution:

Plug-in test: $\frac{0}{0}$ type

$$
\begin{aligned}
\lim_{x\to 3} \frac{x-3}{x^2-4x+3} &= \lim_{x\to 3} \frac{(x-3)}{(x-3)(x-1)} \\
&= \lim_{x\to 3} \tfrac{1}{x-1} \\
&= \lim_{x\to 3} \tfrac{1}{3-1} = \tfrac{1}{2}
\end{aligned}
$$

Evaluate $\lim_{y\to 4} \dfrac{y^2-16}{\sqrt{y}-2}$ (show work):

Solution:

Plug-in test: $\frac{0}{0}$ type

$$
\begin{aligned}
\lim_{y\to 4} \frac{y^2-16}{\sqrt{y}-2} &= \lim_{y\to 4} \frac{(y^2-16)}{(\sqrt{y}-2)} \frac{(\sqrt{y}+2)}{(\sqrt{y}+2)} \\
&= \lim_{y\to 4} \tfrac{(y^2-16)(\sqrt{y}+2)}{y-4} \\
&= \lim_{y\to 4} \tfrac{(y+4)(y-4))(\sqrt{y}+2)}{y-4} \\
&= (4+4)(\sqrt{4}+2) = 8 \cdot 4 = 32
\end{aligned}
$$

Example: Let $f(x) = \begin{cases} ax + 3, & \text{for } x \geq 1 \\ x^2 - a, & \text{for } x < 1. \end{cases}$

a) Find $\lim_{x \to 1^+} f(x)$.

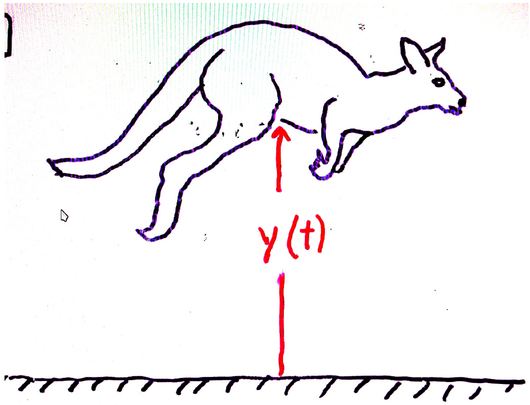
$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} ax + 3$
$= a \cdot 1 + 3 = a + 3.$

b) Find $\lim_{x \to 1^-} f(x)$.

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 - a$
$= 1^2 - a = 1 - a.$

c) Find all a so that $f(x)$ is continuous for all real numbers.

For $x \neq 1$, $f(x)$ is in a small interval around x a polynomial and thus continuous for all a.
For $x = 1$, we need $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$ which is
$a + 3 = 1 - a \;\Leftrightarrow\; 2a = -2 \;\Leftrightarrow\; a = -1.$
Also $f(1) = a \cdot 1 + 3 = a + 3 = \lim_{x \to 1^+} f(x)$.
So, $f(x)$ is exactly for $a = -1$ continuous for all real numbers.



Example:

The height of a Kangaroo (measured in feet) at the time t (measured in seconds) is given by $y = y(t) = 20\,t - 16\,t^2$.

a) Where is the Kangaroo at time $t = 0$?

$y(0) = 0$, on the ground.

b) How high did the Kangaroo jump?

c) When did the Kangaroo reach the maximum height?

We write the parabola $y(t)$ in vertex-point form:

$y(t) = -16(t - 5/8)^2 + 25/4$. Thus the vertex is $(5/8, 25/4)$.

Therefore the maximum height is $25/4$ feet which was reached after $5/8$ seconds.

d) What was is the vertical velocity at that time?

The tangent line to the curve $y(t)$ through the vertex is parallel to the t-axis, i.e. it has slope 0. Thus the vertical velocity is 0 ft./sec.