Calculus I - Lecture 4 - Limits C

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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Section 2.5 — Calculating Limits Algebraically

There are two main types of limits we generally encounter in **Calculus I:**

1. **Plug-in Types**

   Generally this is the case if there is no denominator approaching 0.

2. $\frac{0}{0}$ **type Limits**

   These are the most important types in Calculus.

   Recall, $\frac{0}{0}$ is an undefined quantity, so plug-in fails.

Example: (Plug-in Type)

Evaluate $\displaystyle\lim_{x\longrightarrow\pi}\frac{\sqrt{3x^2-1}}{\sin^2 x-\cos(2x)}$.

Solution:

$$\lim_{x\longrightarrow\pi}\frac{\sqrt{3x^2-1}}{\sin^2 x-\cos(2x)}$$

$$=\frac{\sqrt{3\pi^2-1}}{\sin^2\pi-\cos(2\pi)}=\frac{\sqrt{3\pi^2-1}}{0-1}=-\sqrt{3\pi^2-1}$$

The function $f(x)=\frac{\sqrt{3x^2-1}}{\sin^2 x-\cos(2x)}$ is defined and continuous near $x=\pi$ and so

$$\lim_{x\longrightarrow\pi} f(x)=f(\pi).$$

Example: ($\frac{0}{0}$ type)

Evaluate $\lim\limits_{x \to 2} \dfrac{2x^2 - 4x}{x - 2}$.

Solution:

Always start any limit problem with a plug-in test.

Plug-in: $\dfrac{2 \cdot 2^2 - 4 \cdot 2}{2 - 2} = \dfrac{0}{0}$

Plug-in method **does not work**

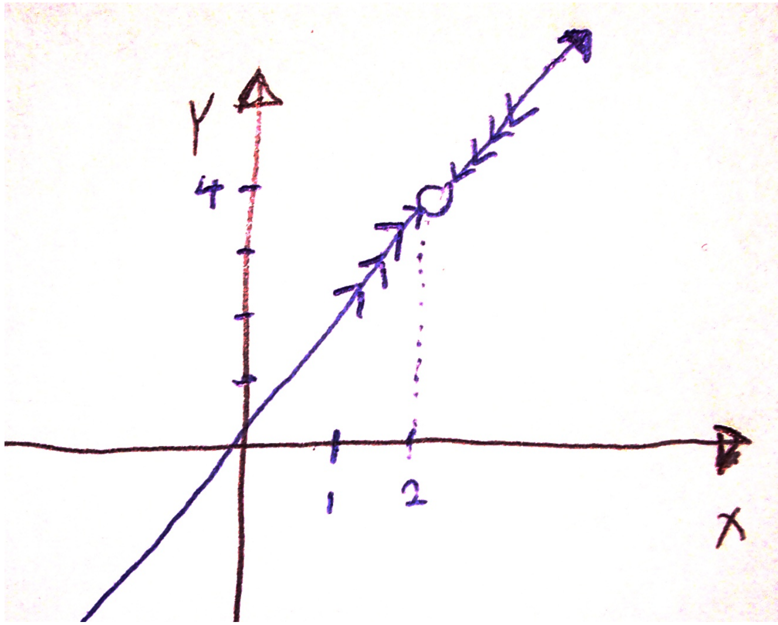
$$\lim_{x \to 2} \frac{2x^2 - 4x}{x - 2} = \lim_{x \to 2} \frac{2x(x-2)}{(x-2)}$$

$$= \lim_{x \to 2} \frac{2x}{1} \quad (x \neq 2, \text{ so } (x-2) \text{ cancels out})$$

$$= \frac{2 \cdot 2}{1} = 4.$$

Graphical interpretation of the limit in previous example

$y = \frac{2x^2-4x}{x-2} = 2x$ (for $x \neq 2$)



Three tricks for $\frac{0}{0}$ type limits

1. Factor top and bottom and cancel the factor approaching 0.
2. Do some algebraic simplification first.
3. Use conjugate trick.

Trick 1 we did in the previous example.

Trick 2: Algebraic simplifications first

Problem: Compute $\lim_{x\to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x-3}$.

Solution:

Plug-in Test: $\frac{\frac{1}{3}-\frac{1}{3}}{3-3} = \frac{0}{0}$ type

$$\lim_{x\to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x-3} = \lim_{x\to 3} \frac{\frac{3}{3}\frac{1}{x} - \frac{1}{3}\frac{x}{x}}{x-3}$$

$$= \lim_{x\to 3} \frac{\frac{3-x}{3x}}{\frac{x-3}{1}}$$

$$= \lim_{x\to 3} \frac{3-x}{3x} \cdot \frac{1}{x-3}$$

$$= \lim_{x\to 3} \frac{-(x-3)}{3x} \cdot \frac{1}{x-3}$$

$$= \lim_{x\to 3} \frac{-1}{3x} = -\frac{1}{9}.$$

Note: $\frac{0}{0}$ type limits can come out to equal any number or D.N.E. (e.g. infinite limits $\pm\infty$ or left-hand and ride-hand limits are different).

$\lim_{x\to 0} \dfrac{x}{x^3} = \lim_{x\to 0} \dfrac{1}{x^2} = +\infty$.

Limit D.N.E. but there is an **infinite limit**.

Trick 3: Conjugation

Recall: The conjugate of $A + \sqrt{B}$ is $A - \sqrt{B}$.

Example: Rationalize the denominator: $\dfrac{1}{2-\sqrt{3}}$.

Solution:

$$\frac{1}{2-\sqrt{3}} = \frac{1}{2-\sqrt{3}} \cdot \frac{2+\sqrt{3}}{2+\sqrt{3}}$$

$$= \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3}.$$

Example: Find $\lim_{x\to 3} \dfrac{\sqrt{x+1}-2}{x-3}$.

Solution:

$$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}$$

$$= \lim_{x\to 3} \frac{(x+1)-4-2\sqrt{x+1}+2\sqrt{x+1}}{(x-3)(\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3} \frac{(x-3)}{(x-3)(\sqrt{x+1}+2)}$$

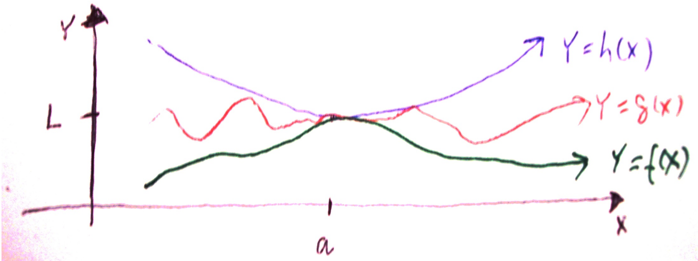
$$= \frac{1}{\sqrt{3+1}+2} = \frac{1}{2+2} = \frac{1}{4}$$

Section 2.6 — Squeeze Theorem and Trigonometric Limits

Theorem (Squeeze Theorem)

Suppose $f(x) \leq g(x) \leq h(h)$ *on an interval containing a and that*

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x).$$



Conclusion: $\lim_{x\to a} g(x) = L.$

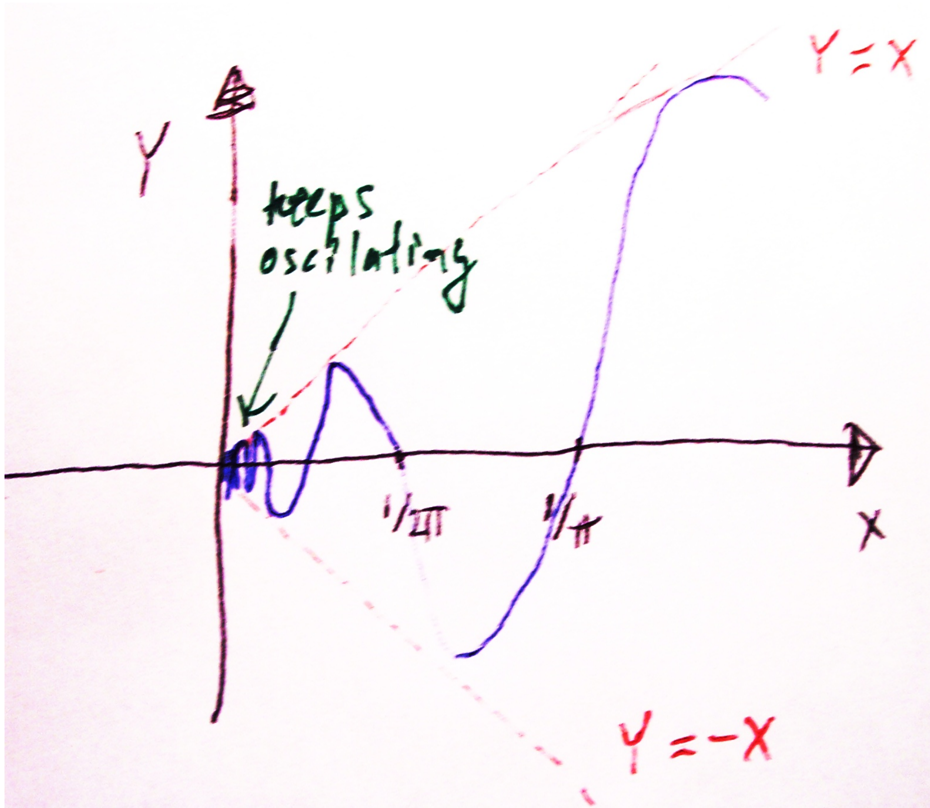
Example: Find $\lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right)$.

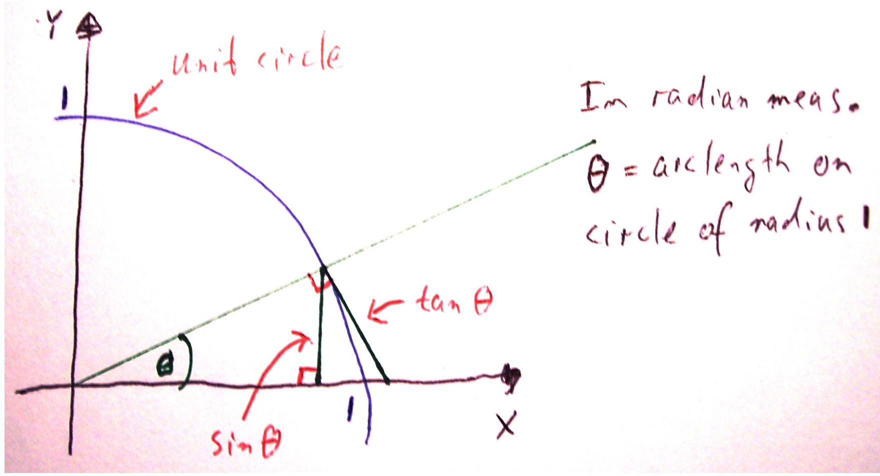
Solution:

$$-x \leq x\sin(\tfrac{1}{x}) \leq x \quad (x > 0)$$

$\lim_{x\to 0} -x = 0 \qquad \lim_{x\to 0} x = 0$

Thus, by the Squeeze Theorem, $\lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right) = 0$


Y = X
Y
keeps oscillating
1/2π
1/π
X
Y = -X



From the diagram: $\sin\theta < \theta < \tan\theta$

equivalent to: $\frac{\sin\theta}{\theta} < 1$ and $\theta < \frac{\sin\theta}{\cos\theta}$, so $\frac{\sin\theta}{\theta} > \cos\theta$

Thus: $$\cos\theta < \frac{\sin\theta}{\theta} < 1$$ ("sandwich")

Apply Squeeze Theorem: $\lim_{\theta\to 0}\cos\theta = \cos(0) = 1$

$\lim_{\theta\to 0} 1 = 1$

and we deduce:

Theorem (Basic Trigonometric Limit)

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

Example: Use the basic trigonometric limit to evaluate the following limits.

a) $\lim_{x\to 0} \dfrac{\sin(4x)}{x}$ $\quad\frac{0}{0}$ type

$= \lim_{x\to 0} \dfrac{4\sin(4x)}{4x}$

$= 4 \cdot 1 = 4$ $\quad$ (basic trig. limit)

b) $\lim_{x\to 0} \dfrac{\sin(2x)}{\sin(5x)}$ $\quad\frac{0}{0}$ type

$= \lim_{x\to 0} \dfrac{\sin(2x)/x}{\sin(5x)/x}$

$= \dfrac{\lim_{x\to 0} \frac{2\sin(2x)}{2x}}{\lim_{x\to 0} \frac{5\sin(5x)}{5x}} = \dfrac{2 \cdot 1}{5 \cdot 1} = \dfrac{2}{5}$

c) $\lim_{x\to 0} \frac{\tan(4x)}{7x}$ $\quad \frac{0}{0}$ type

$$= \lim_{x\to 0} \frac{\sin(4x)}{\cos(4x)\cdot 7x}$$

$$= \lim_{x\to 0} \frac{4\sin(4x)}{4x} \cdot \lim_{x\to 0} \frac{1}{7\cos(4x)}$$

$$= 4 \cdot 1 \cdot \frac{1}{7\cdot\cos(0)} = \frac{4}{7}$$

d) $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$ $\quad \frac{0}{0}$ type

$$= \lim_{x\to 0} \frac{(1-\cos(x))(1+\cos(x))}{x^2 \quad (1+\cos(x))}$$

$$= \lim_{x\to 0} \frac{1-\cos^2(x)}{x^2(1+\cos(x))} = \lim_{x\to 0} \frac{\sin^2(x)}{x^2(1+\cos(x))}$$

$$= \lim_{x\to 0} \left(\frac{\sin(x)}{x}\right)^2 \lim_{x\to 0} \frac{1}{1+\cos(x)}$$

$$= 1^2 \cdot \frac{1}{1+1} = \frac{1}{2}$$