Calculus I - Lecture 3 - Limits B

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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Section 2.3 — Limit Laws

Last lecture, we learned how to evaluate a limit graphically and numerically.

Over the next few sections, we learn how to evaluate limits using:

1. limit laws
2. continuity to evaluate limits by "plugging in a"
3. algrabraic manipulations to resolve $\frac{0}{0}$ type limts.
4. properties of trigonometric functions

Theorem (Basic Limit Laws)

Suppose that $\lim\limits_{x\to a} f(x)$ *and* $\lim\limits_{x\to a} g(x)$ *exist. Then:*

(1) $\lim\limits_{x\to a} (f(x) \pm g(x)) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$

(2) $\lim\limits_{x\to a} c\, f(x) = c \cdot \lim\limits_{x\to a} f(x)$

(3) $\lim\limits_{x\to a} (f(x) \cdot g(x)) = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x)$

(4) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ *(if* $\lim\limits_{x\to a} g(x) \neq 0$*)*

(5) $\lim\limits_{x\to a} f(x)^n = \left(\lim\limits_{x\to a} f(x)\right)^n$

(6) $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$ *(if* $\lim\limits_{x\to a} f(x) \geq 0$ *when* n *even)*

Example: Evaluate using limit laws:

a) $\lim_{x \to 1} (3x^2 - 5x)$ b) $\lim_{x \to 3} \sqrt{\frac{x^2+11}{x-1}}$

Solution:

$$a)\ \lim_{x \to 1} (3x^2 - 5x) = \lim_{x \to 1} 3x^2 - \lim_{x \to 1} 5x \quad \text{by (1)}$$

$$= 3 \lim_{x \to 1} x^2 - 5 \lim_{x \to 1} x \quad \text{by (2)}$$

$$= 3(\lim_{x \to 1} x)^2 - 5 \lim_{x \to 1} x \quad \text{by (5)}$$

$$= 3 \cdot 1^2 - 5 \cdot 1 = 3 - 5 = -2.$$

$$b)\ \lim_{x \to 3} \sqrt{\frac{x^2+11}{x-1}} = \sqrt{\lim_{x \to 3} \frac{x^2+11}{x-1}} \quad \text{by (6)}$$

$$= \sqrt{\frac{\lim_{x \to 3}(x^2+11)}{\lim_{x \to 3}(x-1)}} = \sqrt{\frac{\lim_{x \to 3} x^2 + \lim_{x \to 3} 11}{\lim_{x \to 3} x - \lim_{x \to 3} 1}}$$

$$= \sqrt{\frac{3^2+11}{3-1}} = \sqrt{\frac{20}{2}} = \sqrt{10}$$

Example: Given that $\lim_{x \to 5} f(x) = 3$ and $\lim_{x \to 5} g(x) = 7$ find $\lim_{x \to 5} \dfrac{f(x)^2}{g(x) - 1}$.

Solution:

$$\lim_{x \to 5} \frac{f(x)^2}{g(x) - 1} = \frac{\lim_{x \to 5} f(x)^2}{\lim_{x \to 5} (g(x) - 1)} = \frac{(\lim_{x \to 5} f(x))^2}{\lim_{x \to 5} (g(x) - 1)}$$

$$= \frac{3^2}{7 - 1} = \frac{9}{6} = \frac{3}{2}$$

Example: Assuming $\lim_{x \to 0} \dfrac{f(x)}{x} = 1$ compute $\lim_{x \to 0} \dfrac{x^2 + 3}{x} \cdot \dfrac{f(x)}{1 - x}$.

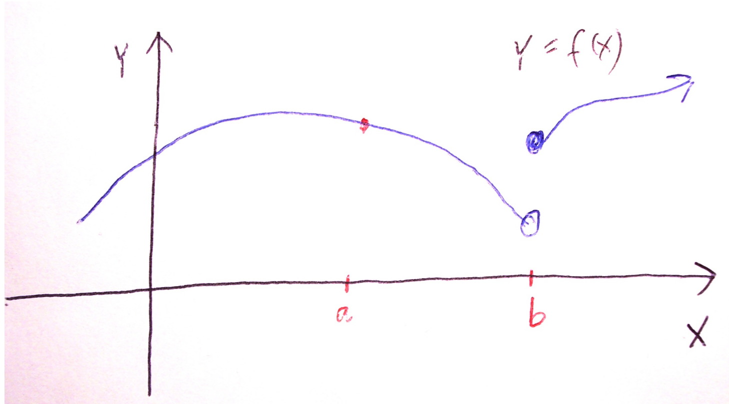
Solution:

$$\lim_{x \to 0} \frac{x^2 + 3}{x} \cdot \frac{f(x)}{1 - x} = \lim_{x \to 0} \frac{f(x)}{x} \cdot \frac{x^2 + 3}{1 - x} = \lim_{x \to 0} \frac{f(x)}{x} \cdot \lim_{x \to 0} \frac{x^2 + 3}{1 - x}$$

$$= 1 \cdot \frac{\lim_{x \to 0}(x^2 + 3)}{\lim_{x \to 0}(1 - x)} = 1 \cdot \frac{0 + 3}{1 - 0} = 3$$

Section 2.4 — Continuity

Intuitive Idea: $f(x)$ is **continuous** at $x = a$ if the graph of $f(x)$ is connected at $x = a$.



In the above graph $f(x)$ is continuous at every point except $x = b$ (discontinuity).

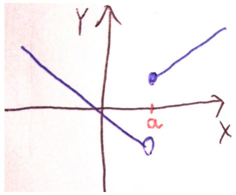
Definition: $f(x)$ is continuous at $x = a$ if

1. $f(a)$ is defined,
2. $\lim_{x \to a} f(x)$ exists, and
3. $\lim_{x \to a} f(x) = f(a)$.

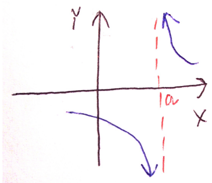
In other words, the limit can be evaluated by plugging in $x = a$.

A point where $f(x)$ is not continuous is called a **discontinuity.**

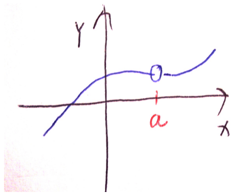
4 **types of discontinuities:**

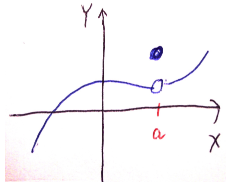
1. Jump: 

   $\lim_{x\longrightarrow a^+} f(x) \neq \lim_{x\longrightarrow a^+} f(x)$, so $\lim_{x\longrightarrow a} f(x)$ D.N.E.

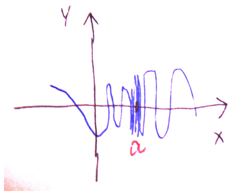
2. Infinite: 

   $\lim_{x\longrightarrow a} f(x)$ D.N.E.

3. Removable: 

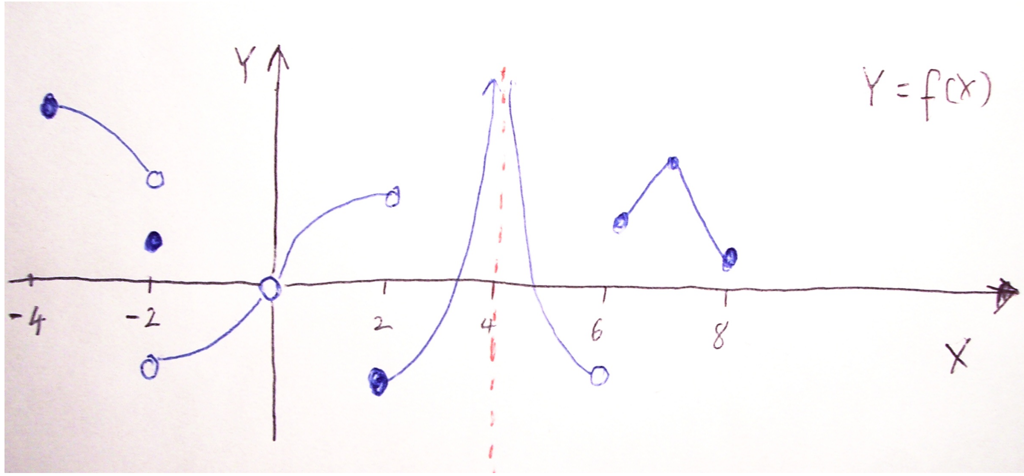
   $f(a)$ not defined $\qquad \lim_{x\longrightarrow a} f(x) \neq f(a)$

4. Oscillating: 

   $\lim_{x\longrightarrow a} f(x)$ D.N.E.

Example: a) Find the discontinuities of $f(x)$.

b) State the intervals where $f(x)$ is continuous.



Solution:

a) Discontinuities:

- $x = -2$ (jump)
- $x = 0$ (not defined)
- $x = 2$ (jump)
- $x = 4$ (infinite)
- $x = 6$ (jump)

b) Domain of $f(x) \subseteq [-4, 8]$. Intervals where $f(x)$ is continuous:

$[-4, -2)$, $(-2, 0)$, $(0, 2)$, $(2, 4)$, $(4, 6)$, $(6, 8]$.

One sided continuity:

1. $f(x)$ is **continuous from the right** at $x = a$ if

   $$\lim_{x \to a^+} f(x) = f(a)$$

2. $f(x)$ is **continuous from the left** at $x = a$ if

   $$\lim_{x \to a^-} f(x) = f(a)$$

Example: In previous example:

a) Is $f(x)$ right continuous at $x = 2$?

Yes: $\lim_{x \to 2^+} f(x) = -2, \qquad f(2) = -2 \qquad$ (both are equal).

b) Is $f(x)$ left continuous at $x = 2$?

No: $\lim_{x \to 2^-} f(x) = 2, \qquad f(2) = -2$ (both are different).

c) Is $f(x)$ left continuous at $x = 0$?

No: $f(0)$ is not defined.

Example: Determine whether $f(x) = \begin{cases} x^2 - 2, & \text{if } x > 2 \\ 3 - x, & \text{if } x \leq 2 \end{cases}$ is

a) right continuous at $x = 2$, b) left continuous at $x = 2$,
c) continuous at $x = 2$.

Solution:

a) $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 - 2) = 2^2 - 2 = 2.$
$f(2) = 3 - 2 = 1.$
Since $\lim_{x \to 2^+} f(x) \neq f(2)$, $f(x)$ **is not** right continuous at $x = 2$.

b) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (3 - x) = 3 - 2 = 1.$
$f(2) = 3 - 2 = 1.$
Since $\lim_{x \to 2^-} f(x) = f(2)$, $f(x)$ **is** left continuous at $x = 2$.

c) Since $\lim_{x \to 2^+} f(x) \neq \lim_{x \to 2^-} f(x)$, $f(x)$ **is not** continuous at $x = 2$.

Example: Determine the value(s) of c so that $f(x)$ is continuous at $x = 2$ where $f(x) = \begin{cases} x^2 - 3, & x \geq 2, \\ 2x - c, & x < 2. \end{cases}$

Solution: We compute:

- $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 - 3) = 2^2 - 3 = 1$
- $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x - c) = 2 \cdot 2 - c = 4 - c$
- $f(2) = 2^2 - 3 = 1$

For $f(x)$ to be continuous we need

$$\begin{aligned} & \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = f(2) \\ \Leftrightarrow \quad & 1 = 4 - c = 1 \\ \Leftrightarrow \quad & c = 4 - 1 = 3. \end{aligned}$$

Theorem

The following functions are continuous:

- $f(x) = x^n$ *for integer numbers* n *everywhere on its domain;*
- $f(x) = x^{1/n}$ *for natural numbers* n *everywhere on its domain;*
- $f(x) = b^x$ *for* $b > 0$ *on the whole real line;*
- $f(x) = \log_b x$ *for* $b > 0$ *and* $x > 0$*;*
- $f(x) = \sin x$ *and* $f(x) = \cos x$ *on the whole real line.*

Theorem

Let $f(x)$ *and* $g(x)$ *be continuous at* $x = c$*. Then also the following functions are continuous at* $x = c$*:*

$$f(x) \pm g(x); \qquad f(x) \cdot g(x); \qquad \frac{f(x)}{g(x)} \text{ if } g(c) \neq 0.$$

Theorem

If $g(x)$ *is continuous at* $x = c$ *and* $f(x)$ *is continuous at* $x = g(c)$*, then the composite function* $F(x) = f(g(x))$ *is continuous at* $x = c$*.*

Example Compute $\lim_{x->0} \frac{\cos(x^2)}{1-x^2}$.

Solution:

x^2 (power function) and $\cos(x)$ (cosine function) are everywhere defined and continuous.
$\Rightarrow \cos(x^2)$ is everywhere defined and continuous (composite function).

1 (constant function) and x^2 (power function) are everywhere defined and continuous.
$\Rightarrow 1 - x^2$ is everywhere defined and continuous (difference).

$1 - x^2 = 0 \Leftrightarrow x^2 = 1 \Leftrightarrow x = -1$ or $x = 1$.
$\Rightarrow f(x) := \frac{\cos(x^2)}{1-x^2}$ (quotient function) is defined and continuous for all $x \neq \pm 1$.

Since $f(x)$ is continuous at $x = 0$ we have

$$\lim_{x->0} f(x) = f(0) = \frac{\cos(0^2)}{1-0^2} = \frac{\cos(0)}{1} = 1.$$