Calculus I - Lecture 26 - Area and Volume

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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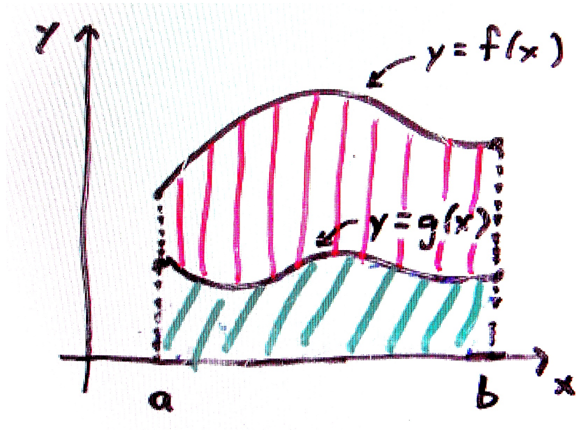
Section 6.1 – Area between curves



Area below $f(x)$: Red + Green

Area below $g(x)$: Green

Let $A =$ area of region between the curves $y = f(x)$ and $y = g(x)$ over $[a, b]$ (Red Area).

Assume $f(x) \geq g(x)$ on $[a, b]$.

$$\underbrace{A}_{\text{Red}} = \underbrace{\int_a^b f(x)\,dx}_{\text{Red + Green}} - \underbrace{\int_a^b g(x)\,dx}_{\text{Green}} = \int_a^b (f(x) - g(x))\,dx$$

The calculation is usually simpler if you subtract the functions first.

Example: Find the area of the region trapped between the curves $y = x^2$ and $y = 6 - x$.

Solution:

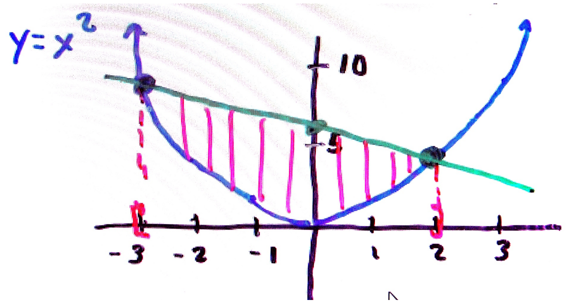
Intersection points: $x^2 = 6 - x \;\Leftrightarrow\; x^2 + x - 6 = 0 \;\Leftrightarrow$
$(x+3)(x-2) = 0 \;\Leftrightarrow\; x = -3$ or $x = 2$

Thus x runs from -3 to 2. Furthermore, we see $6 - x \geq x^2$ in that range. Therefore:

$$A = \int_{-3}^{2} (\text{Top} - \text{Bottom})\, dx = \int_{-3}^{2} \left((6-x) - x^2\right) dx$$

$$= \left(6x - \frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_{-3}^{2} = (12 - 2 - \frac{8}{3}) - (-18 - \frac{9}{2} + 9)$$

$$= \frac{125}{6}$$



The same method works if either $f(x)$ or $g(x)$ is negative or $g(x) \geq f(x)$. In the later case integrate $|f(x) - g(x)| = g(x) - f(x)$.

Example: Find the area of the region trapped between $y = -2x$ and $y = 6 - x^2$. Just set up the integral.

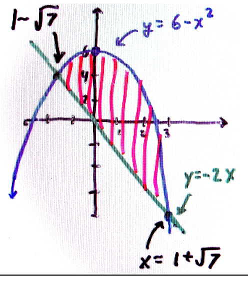
Solution:

Intersection points: $-2x = 6 - x^2 \Leftrightarrow x^2 - 2x - 6 = 0 \Leftrightarrow$

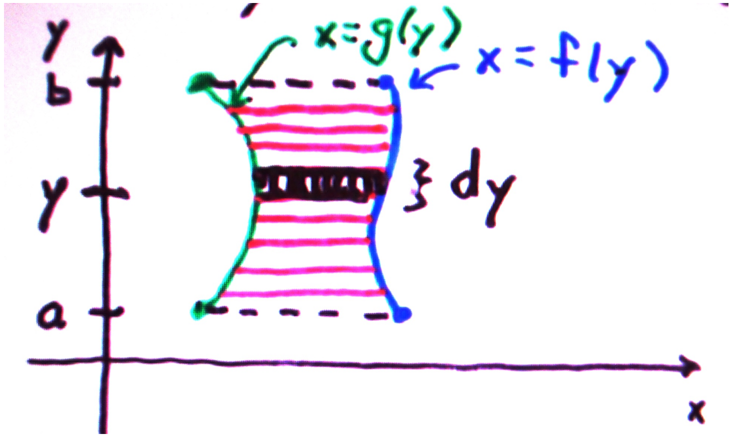
$$x = \frac{2 \pm \sqrt{4 - 4 \cdot (-6)}}{2} = \frac{2 \pm \sqrt{28}}{2} = 1 \pm \sqrt{7}$$

Thus x runs from $1 - \sqrt{7}$ to $1 + \sqrt{7}$. Furthermore, we have $6 - x^2 \geq -2x$ in that range. Therefore:

$$A = \int_{1-\sqrt{7}}^{1+\sqrt{7}} (\text{Top} - \text{Bottom})\, dx = \int_{1-\sqrt{7}}^{1+\sqrt{7}} (6 - x^2 - (-2x))\, dx$$



Areas by horizontal slices



$A =$ area of region between $x = f(y)$ and $x = g(y)$ over the y-interval $[a, b]$.

Assume $g(y) \leq f(y)$.

$$dA = \text{area of thin horizontal slice of width } dy$$
$$\text{length} \cdot dy = \big(f(y) - g(y)\big) \cdot dy$$

$$A = \int_a^b da = \int_a^b \big(f(y) - g(y)\big) \cdot dy$$

Same formula with y instead of x.

Example: Find the area of the region bounded by $x = y^2 - 4y$ and the y-axis.

Solution:

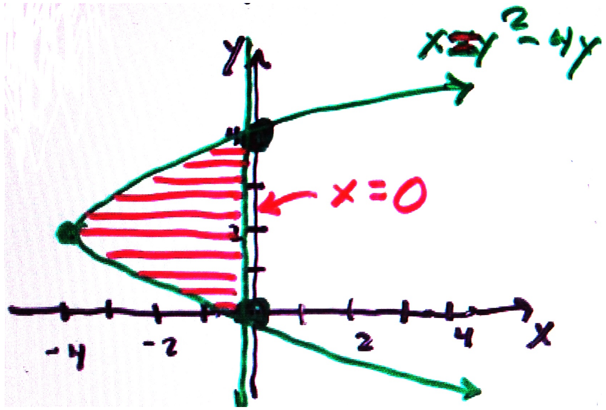
Intersection between curve and y-axis:

$$x = y^2 - 4y = 0 \iff y(y-4) = 0 \iff y = 0 \text{ or } y = 4.$$

Thus:

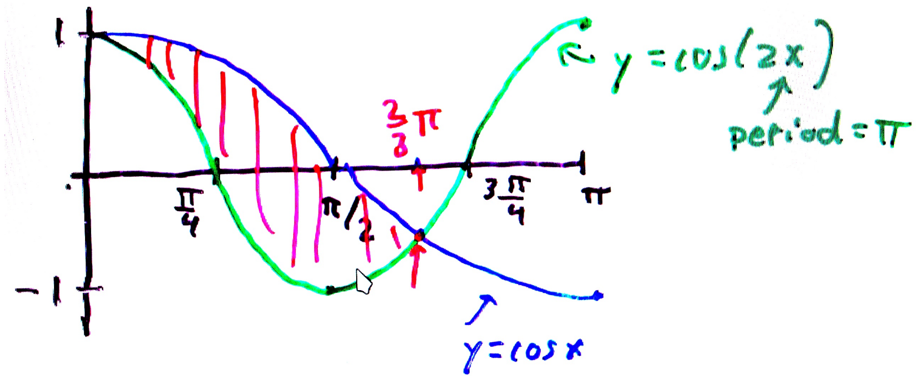
$$\int_0^4 |y^2 - 4y|\, dy = \int_0^4 -(y^2 - 4y)\, dy$$

$$= -\frac{y^3}{3} + 4\frac{y^2}{2}\Big|_0^4 = -\frac{64}{3} + 2 \cdot 16 = \frac{32}{3}$$



Example: Find the area of the region between $y = \cos x$ and $y = \cos(2x)$ over $[0, \frac{2}{3}\pi]$.

Solution:



We note that 0 and $\frac{2}{3}\pi$ are the two intersection points in $[0, 2\pi)$.

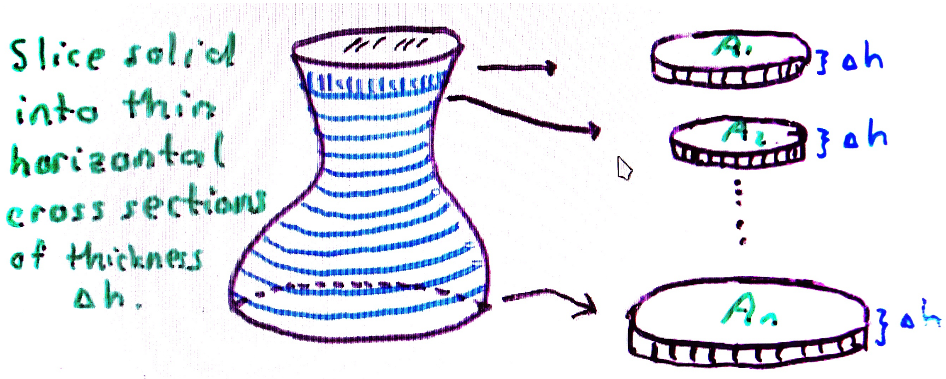
$$A = \int_0^{\frac{2}{3}\pi} (\cos x - \cos(2x))\, dx = \left(\sin x - \sin(2x) \cdot \frac{1}{2}\right)\Big|_0^{\frac{2}{3}\pi}$$

$$= \sin(\frac{2}{3}\pi) - \frac{1}{2}\sin(2 \cdot \frac{2}{3}\pi) = \frac{\sqrt{3}}{2} - \frac{1}{2}(-\frac{\sqrt{3}}{2})$$

$$= \frac{3\sqrt{3}}{4}$$

Section 6.2 – Volumes by Cross-Sections

Basic Problem: Determine the volume V of a solid like the one below.



Let A_i = cross sectional area of the j^{th} slice.

V_j = volume of the j^{th} slice $\approx A_j \cdot \Delta h$.

$$V = \sum_{j=1}^{n} V_j \approx \sum_{j=1}^{n} A_j \, \Delta h$$

$$V = \lim_{\Delta h \to 0} \sum_{j=1}^{n} A_j \, \Delta h = \int_a^b A \, dh$$

Where $A = A(h)$ is the **cross-sectional area** and h runs from a to b.

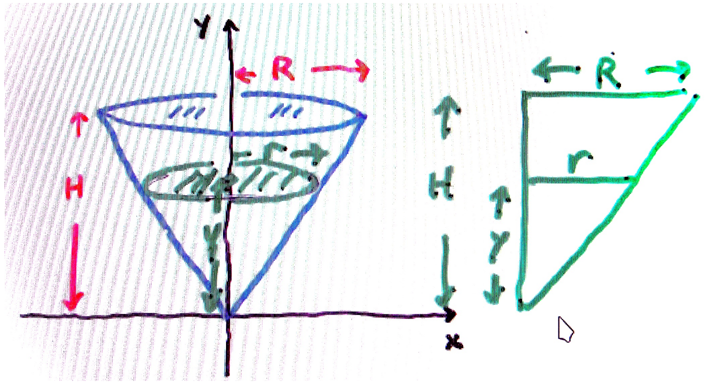
Theorem (Volume by Cross-Section Formula)

$$V = \int_a^b A(y)\,dy$$

where $A(y)$ is the area of a cross-section perpendicular to the y-axis.

Example: Find the volume of a cone of height H and radius R.

Solution:



The similar triangles give $\frac{r}{y} = \frac{R}{H}$ or $r = \frac{R}{H}y$.

$$A(y) = \text{area of circle of radius } r$$

$$= \pi r^2 = \pi \left(\frac{R}{H}y\right)^2 = \pi \frac{R^2}{H^2}y^2$$

$$V = \int_0^H A(y)\,dy = \int_0^H \pi \frac{R^2}{H^2}\,y^2\,dy$$

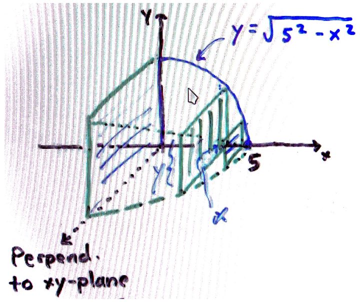
$$= \pi \frac{R^2}{H^2}\frac{y^3}{3}\bigg|_0^H = \pi \frac{R^2}{H^2}\frac{H^3}{3} = \frac{1}{3}\pi R^2 H$$

If the cross-sections are perpendicular to the x-axis the formula is:

$$V = \int_a^b A(x)\,dx$$

where $A(x)$ is the area of a cross-section.

Example: Find the volume of a solid whose base is a quarter disk or radius 5 the xy-plane as shown and such that each cross section perpendicular to the x-axis is a square.



Solution: $y = \sqrt{5^2 - x^2} \iff x^2 + y^2 = 5^2$, circle of radius 5.

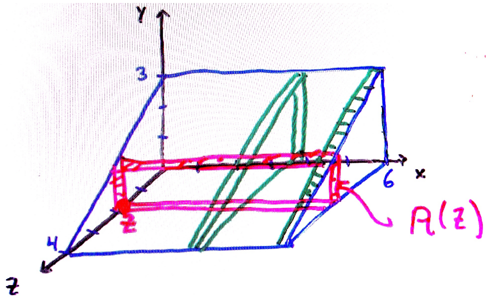
Cross sectional area $A(x) = y^2 = 5^2 - x^2$.

$$V = \int_0^5 A(x)\,dx = \int_0^5 (5^2 - x^2)\,dx$$

$$= \left(25x - \frac{x^3}{3}\right)\Bigg|_0^5 = 24 \cdot 5 - \frac{5^3}{3} = 125 \cdot \frac{2}{3} = \frac{250}{3}.$$

For some regions, the volume can be calculated in more than one way using cross sections.

Example: Find the volume of the wedge



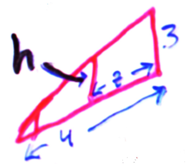
Solution: a) Using cross section perpendicular to x-axis:

Area of green triangle $= \frac{1}{2} \cdot 4 \cdot 3 = 6.$

$$V = \int_0^6 A(x)\,dx = \int_0^6 dx = 6x\Big|_0^6 = 36$$

Solution: b) Using cross section perpendicular to z-axis:

Area of red rectangle $= 6 \cdot h = 6 \cdot \frac{3}{4}(4 - z) = \frac{9}{2}(4 - z).$



$$V = \int_0^4 A(x)\,dx = \int_0^4 \tfrac{9}{2}(4-z)\,dx = \tfrac{9}{2}(4z - \tfrac{1}{2}z^2)\Big|_0^4 = \tfrac{9}{2}(16-8) = 36$$