Calculus I - Lecture 25
Net change as Integral of a Rate

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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Section 5.7 - Miscellaneous Integrals

From our table of derivatives we obtain the following integrals:

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int b^x\,dx = \frac{1}{\ln b}\cdot b^x + C$$

$$\int \frac{dx}{1+x^2} = \arctan x + C = \tan^{-1} x + C$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C = \sin^{-1} x + C$$

$$\int \frac{dx}{|x|\sqrt{x^2-1}} = \text{arcsec} x + C = \sec^{-1} x + C$$

Memorize!

The last integral will be done in Calculus II

Example: Evaluate $\displaystyle\int \frac{dx}{9+4x^2}$.

Solution:

We have $9+4x^2$ but want $1+u^2$.

$$9+4x^2 = 9(1+\tfrac{4}{9}x^2) = 9(1+(\tfrac{2}{3}x)^2).$$

Let $u = \dfrac{2}{3}x$. Then $du = \dfrac{2}{3}\,dx$ or $\dfrac{3}{2}\,du = dx$.

$$\begin{aligned}
\int \frac{dx}{9+4x^2} &= \int \frac{dx}{9(1+(\frac{2}{3}x)^2)} \\
&= \frac{1}{9}\int \frac{dx}{1+(\frac{2}{3})^2} \\
&= \frac{1}{9}\int \frac{\frac{3}{2}\,du}{1+u^2} \\
&= \frac{3}{18}\arctan u + C \\
&= \frac{1}{6}\arctan\left(\frac{2}{3}x\right) + C
\end{aligned}$$

Last week we saw the Fundamental Theorem of Calculus:

Theorem (Fundamental Theorem of Calculus I)

Let $f(x)$ be a continuous function on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b := F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$.

Today we will look at an important **application** of this formula.

Let:

t = time
$s = s(t)$ = quantity we are measuring as a function of t.
$r(t) = s'(t)$ = rate of change of s with respect to t.

Theorem (Net change formula)

$$\underbrace{\int_a^b r(t)\,dt}_{\text{Integral of the rate of change } r} = \underbrace{s(b) - s(a)}_{\text{Net change in } s \text{ over } [a,b]}$$

Why does this formula hold?

Note, $s(t)$ is an antiderivative of $r(t)$ since $s'(t) = r(t)$.
Thus by the F.T.C.:

$$\int_a^b r(t)\,dt = s(t)\big|_a^b = s(b) - s(a).$$

Example: Water is flowing into a tank at a rate of $r(t) = t^2$ $\mathrm{ft}^3/\mathrm{min}$. How much water flows into the tank over the time interval 1 min. to 5 min.?

Solution:

Let $V = V(t)$ be the volume of the water in the tank.

Then $V'(t) = r(t) = t^2$.

$$\underbrace{V(5) - V(1)}_{\text{Net change in volume}} = \int_1^5 r(t)\,dt = \int_1^5 t^2\,dt$$

$$= \left.\frac{t^3}{3}\right|_1^5 = \frac{5^3}{3} - \frac{1^3}{3}$$

$$= \frac{125-1}{3} = \frac{124}{3}$$

From min. 1 to min. 5 there flows a total of $\frac{124}{3}$ ft^3 into the tank.

Example: The power used by an appliance is given by

$$P(t) = 2 + \sin\left(\frac{\pi t}{12}\right) \quad \text{joule/hr.}$$

How much energy is consumed during a one day period, $0 \le t \le 24$?

Solution:

Let $E(t)$ be the energy consumed (in joule).
Then $E'(t) = P(t)$ is the rate of energy consumed (in joule/hr).

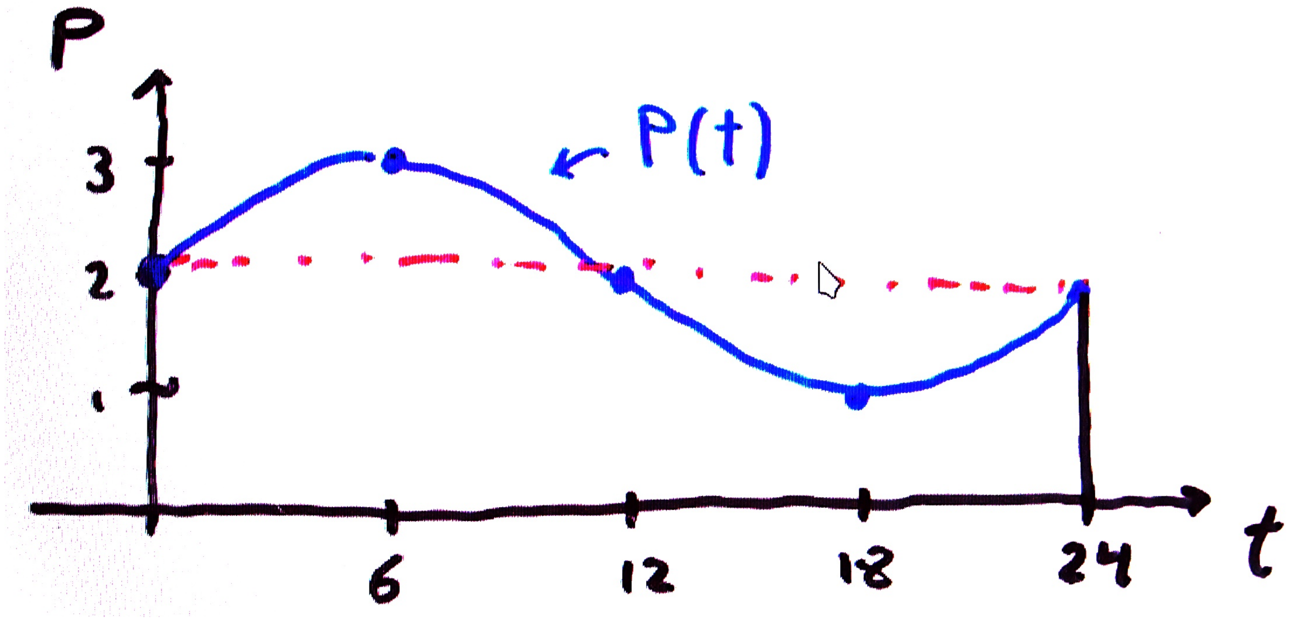
$$E(24) - E(0) = \int_0^{24} P(t)\,dt = \int_0^{24} \left(2 + \sin\left(\frac{\pi t}{12}\right)\right)\,dt$$

$$= \left(2t - \cos\left(\frac{\pi t}{12}\right)\frac{12}{\pi}\right)\Bigg|_0^{24}$$

$$= \left[2 \cdot 24 - \frac{12}{\pi}\cos(2\pi)\right] - \left[2 \cdot 0 - \frac{12}{\pi}\cos(0)\right]$$

$$= \left[48 - \frac{12}{\pi}\right] - \left[0 - \frac{12}{\pi}\right] = 48$$

During one day the appliance consumes 48 joules.



P
P(t)
3
2
1
6
12
18
24
t

Motion along a straight line

An object moves along the s-axis. Let

$s(t)$ be the **position** at time t

$v(t) = s'(t)$ be the **velocity** at time t

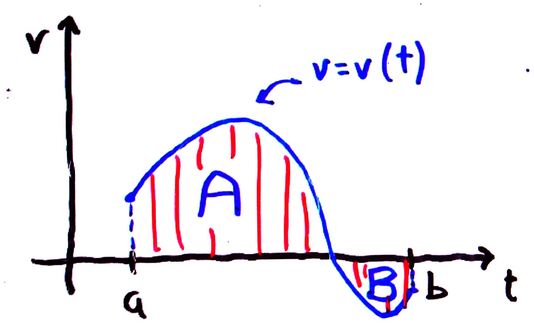
$|v(t)|$ be the **speed** at time t.

Then:

Change in position over time interval $[a, b]$ $= s(b) - s(a) = \int_a^b v(t)\, dt$

Total distance traveled over time interval $[a, b]$ $= \int_a^b |v(t)|\, dt$

(This is what an odometer is measuring)



Displacement $= A - B$

Total Distance $= A + B$

A moving right, B moving left

Example: The velocity (in m/sec) of an object is given by $v(t) = t^2 - 2t$. Find the displacement and total distance traveled over the interval $0 \le t \le 5$ (t in sec).

Solution:

a) Displacement $= \int_0^5 (t^2 - 2t)\,dt = \left(\frac{t^3}{3} - t^2\right)\Big|_0^5$

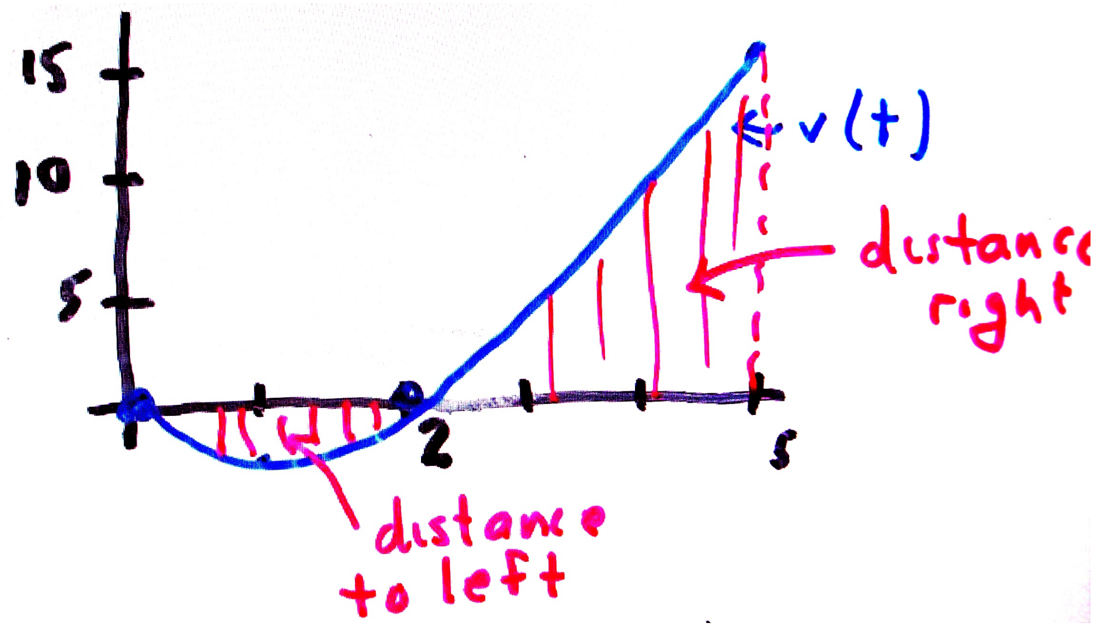
$$= \left(\frac{5^3}{3} - 5^2\right) - (0) = \frac{50}{3}\ m.$$

b) $v(t) = t(t-2)$. Thus $v(t) \le 0$ for $t \in [0, 2]$ and $v(t) \ge 0$ for $t \in [2, 5]$.

Total dist. $= \int_0^5 |t^2 - 2t|\,dt = \int_0^2 -(t^2 - 2t)\,dt + \int_2^5 (t^2 - 2t)\,dt$

$$= \left(-\frac{t^3}{3} + t^2\right)\Big|_0^2 + \left(\frac{t^3}{3} - t^2\right)\Big|_2^5$$

$$= \left(-\frac{8}{3} + 4\right) - 0 + \left(\frac{5^3}{3} - 5^2\right) - \left(\frac{8}{3} - 4\right) = \frac{58}{3}\ m.$$



15
10
5
2
5
v(t)
distance right
distance to left

Example: The acceleration of an object traveling on a straight line is given by $a(t) = 3\,e^{2t}\ \mathrm{m/sec^2}$ where t is the time in seconds. Find the change in velocity over the time interval $[1, 3]$ seconds.

Solution:

$$\text{Change in velocity} = v(3) - v(1) = \int_1^3 a(t)\,dt$$

$$= \int_1^3 3e^{2t}\,dt$$

$$= 3e^{2t} \cdot \frac{1}{2}\Big|_1^3$$

$$= \frac{3}{2}e^6 - \frac{3}{2}e^2$$

$$= \frac{3}{2}e^2(e^4 - 1).$$

The velocity changes by $\frac{3}{2}e^2(e^4 - 1)\ \mathrm{m/sec}$ over the time interval from one 1 to 3 seconds.

Marginal Cost

Let x be the number of items of a product a company makes.

Let $C(x)$ be the cost to make x items.

$C'(x)$ is called the **marginal cost** $\approx$ the cost to produce one more item if currently x items are produced.

$$C'(x) \approx \frac{C(x+1) - C(x)}{1} = C(x+1) - C(x)$$

Example: What is the cost to produce 20 items if the set-up cost is \$1000 and $C'(x) = 100 - 20x + x^2$?

Solution:

$$C(20) - C(0) = \int_0^{20} c'(x)\,dx = \int_0^{20} (100 - 20x + x^2)\,dx$$

$$= \left(100x - 10x^2 + \frac{x^3}{3}\right)\Big|_0^{20}$$

$$= 2000 - 4000 + \frac{8000}{3} = \frac{2000}{3}$$

$$C(20) = \frac{2000}{3} + C(0) = \frac{2000}{3} + 1000 = \frac{5000}{3}.$$

The total cost of producing 20 items is \$1,666.67.