Calculus I - Lecture 24
The Substitution Method

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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Integration by Substitution

One of the goals of Calculus I and II is to develop techniques for evaluating a wide range of indefinite integrals.

Of the 111 integrals on the back cover of the book we can do the first 16 this course. The rest will be done in Calculus II.

Examples from the Table we already know:

$$\begin{aligned}
\int u^n\,du &= \frac{u^{n+1}}{n+1} + C, \quad n \neq -1 \\
\int \frac{du}{u} &= \ln|u| + C \\
\int \sec^2 u\,du &= \tan u + C \\
\int \frac{du}{1+u^2} &= \tan^{-1} u + C
\end{aligned}$$

Each rule for derivatives yields a corresponding rule for integrals.

Chain Rule for derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}x} F(g(x)) \;=\; F'(g(x)) \cdot g'(x)$$

Corresponding integral rule:

$$\int F'(g(x)) \cdot g'(x) \;=\; F(g(x)) + C$$

Example:

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin(x^3) = \cos(x^3) \cdot 3x^2$$

$$\int \cos(x^3) \cdot 3x^2 \, dx \;=\; \sin(x^3) + C$$

We have determined the antiderivative of $\cos(x^3) \cdot 3x^2$.

Substitution Method

Integration by substitution, called *u*-**substitution** is a method of evaluating integrals of the type

$$\int \underbrace{f(g(x))}_{\text{Composite function}} \cdot g'(x)\, dx$$

Four steps:

1. Set $u = g(x)$. Then $\frac{du}{dx} = g'(x)$ or $du = g'(x)\, dx$.
2. Substitute these values of u and du to convert original integral into integral for the new variable u.
3. Compute integral in the new variable u.
4. Replace u by $g(x)$, i.e., express result in the original variable.

Example: Find $\int \cos(x^3) \cdot x^2\, dx$.

Solution:

1. Let $u = x^3$. Then $\dfrac{du}{dx} = 3x^2$ or $du = 3x^2\, dx$.

   Need: $x^2\, dx$. We get $x^2\, dx = \dfrac{1}{3}\, du$. Thus:

2. $$\int \cos(x^3) \cdot x^2\, dx = \int \cos(u) \cdot \frac{1}{3}\, du$$

3. $$= \frac{1}{3} \int \cos u\, du$$

   $$= \frac{1}{3} \sin u + C$$

4. $$= \frac{1}{3} \sin x^3 + C$$

Example: Find $\int x\sqrt{1-x^2}\,dx$.

Solution:

1. Let $u = 1 - x^2$. Then $\dfrac{du}{dx} = -2x$ or $du = -2x\,dx$.

   Need: $x\,dx$. We get $x\,dx = -\dfrac{1}{2}\,du$. Thus:

2. $$\int x\sqrt{1-x^2}\,dx = \int \sqrt{u}\cdot(-\frac{1}{2})\,du$$

3. $$= -\frac{1}{2}\int u^{1/2}\,du$$
   $$= -\frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$$

4. $$= -\frac{1}{3}(1-x^2)^{3/2} + C$$

You may check this by differentiation:

$$\left(-\frac{1}{3}(1-x^2)^{3/2} + C\right)' = -\frac{1}{3}\cdot\frac{3}{2}(1-x^2)^{1/2}\cdot 2x = x\sqrt{1-x^2}$$

Example: Find $\displaystyle\int \frac{\sec^2(3x)}{\tan(3x)}\,dx$.

Solution:

1. Let $u = \tan(3x)$. Then $\dfrac{du}{dx} = \sec^2(3x)\cdot 3$ or $du = 3\sec^2(3x)\,dx$.

   Need: $\sec^2(3x)\,dx$. We get $\sec^2(3x)\,dx = \dfrac{du}{3}$. Thus:

2. $$\int \frac{\sec^2(3x)}{\tan(3x)}\,dx = \int \frac{1}{u}\cdot\frac{du}{3}$$

3. $$= \tfrac{1}{3}\frac{du}{u}$$

$$= \frac{1}{3}\cdot \ln|u| + C$$

4. $$= \frac{1}{3}\ln|\tan(3x)| + C$$

Definite Integral: Two ways to evaluate using u-substitution

1. Find indefinite integral, and plug in original limits. (Old way)
2. Change limits to new variable u. (New way)

Example: Find $\int_0^{\pi/2} \sqrt{\cos x} \sin x \, dx$ the new way.

Solution:

Let $u = \cos x$. Then $du = -\sin x \, dx$ or $-du = \sin x \, \mathrm{d}x$.

Integration bounds:

When $x = 0$ then $u = \cos 0 = 1$
When $x = \pi/2$ then $u = \cos(\pi/2) = 0$

$$\int_0^{\pi/2} \sqrt{\cos x} \sin x \, dx = \int_1^0 \sqrt{u} \, (-1) \, du$$
$$= -\int_1^0 u^{1/2} \, du = -\frac{2}{3} u^{3/2} \Big|_1^0$$
$$= -0 - \left(-\frac{2}{3} \cdot 1^{3/2} \right) = \frac{2}{3}$$

Some tricky u-substitutions

Example: Find $\int x\sqrt{2x+1}\,dx$.

Solution:

Let $u = 2x + 1$. Then $du = 2dx$ or $\frac{du}{2} = dx$.

Also $u = 2x + 1 \;\Leftrightarrow\; u - 1 = 2x \;\Leftrightarrow\; x = \frac{u-1}{2}$

$$\begin{aligned}
\int x\sqrt{2x+1}\,dx &= \int \frac{u-1}{2}\sqrt{u}\cdot\frac{du}{2} \\
&= \frac{1}{4}\int (u-1)u^{1/2}\,du \\
&= \frac{1}{4}\int \left(u^{3/2} - u^{1/2}\right)\,du \\
&= \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] + C \\
&= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C
\end{aligned}$$

Example: Find $\dfrac{dx}{x(\ln x)^2}$.

Solution:

Let $u = \ln x$. Then $du = \dfrac{1}{x}\,dx$

$$\int \frac{dx}{x(\ln x)^2} = \int \frac{1}{u^2}\,du$$

$$= \int u^{-2}\,du$$

$$= \frac{u^{-1}}{-1} + C$$

$$= -\frac{1}{u} + C$$

$$= -\frac{1}{\ln x} + C$$

Section 5.7 - Miscellaneous Integrals

From our table of derivatives we obtain the following integrals:

$$\begin{aligned}
\int \frac{1}{x}\, dx &= \ln|x| + C \\
\int b^x\, dx &= \frac{1}{\ln b} \cdot b^x + C \\
\int \frac{dx}{1+x^2} &= \arctan x + C = \tan^{-1} x + C \\
\int \frac{dx}{\sqrt{1-x^2}} &= \arcsin x + C = \sin^{-1} x + C \\
\int \frac{dx}{|x|\sqrt{x^2-1}} &= \text{arcsec} x + C = \sec^{-1} x + C
\end{aligned}$$

Memorize!

The last integral will be done in Calculus II

Example: Find $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$.

Solution:

One has $9 - x^2 = 9(1 - \frac{1}{9}x^2) = 9(1 - (\frac{x}{3})^2)$.

Let $u = \dfrac{x}{3}$. Then $du = \dfrac{1}{3}\,dx$ or $3\,du = dx$.

$$\begin{aligned}
\int \frac{dx}{\sqrt{9-x^2}} &= \int \frac{dx}{\sqrt{9(1-(\frac{x}{3})^2)}} \\
&= \frac{1}{3}\int \frac{dx}{\sqrt{1-(\frac{x}{3})^2}} \\
&= \frac{1}{3}\int \frac{3\,du}{\sqrt{1-u^2}} \\
&= \arcsin u + C \\
&= \arcsin\left(\frac{x}{3}\right) + C
\end{aligned}$$

Example: Evaluate $\displaystyle\int \frac{dx}{9+4x^2}$.

Solution:

We have $9+4x^2$ but want $1+u^2$.

$$9+4x^2 = 9(1+\tfrac{4}{9}x^2) = 9(1+(\tfrac{2}{3}x)^2).$$

Let $u = \dfrac{2}{3}x$. Then $du = \dfrac{2}{3}\,dx$ or $\dfrac{3}{2}\,du = dx$.

$$\begin{aligned}
\int \frac{dx}{9+4x^2} &= \int \frac{dx}{9(1+(\frac{2}{3}x)^2)} \\
&= \frac{1}{9}\int \frac{dx}{1+(\frac{2}{3})^2} \\
&= \frac{1}{9}\int \frac{\frac{3}{2}\,du}{1+u^2} \\
&= \frac{3}{18}\arctan u + C \\
&= \frac{1}{6}\arctan\left(\frac{2}{3}x\right) + C
\end{aligned}$$

Example: Evaluate $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$.

Solution:

Let $u = e^x$. Then $du = e^x\,dx$.

$$\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{e^x\,dx}{1+(e^x)^2}$$

$$= \int \frac{du}{1+u^2}$$

$$= \arctan u + C$$

$$= \arctan(e^x) + C$$