Calculus I - Lecture 23
Fundamental Theorem of Calculus

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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April 16, 2014

Section 5.3 - Fundamental Theorem of Calculus I

We have seen two types of integrals:

1. **Indefinite:** $\displaystyle\int f(x)\,dx = F(x) + C$

   where $F(x)$ is an antiderivative of $f(x)$.

2. **Definite:** $\displaystyle\int_a^b f(x)\,dx =$ signed area bounded by $f(x)$ over $[a, b]$.

Theorem (Fundamental Theorem of Calculus I)

Let $f(x)$ be a continuous function on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b := F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$.

Note: The result is independent of the chosen antiderivative $F(x)$.

We have **three** ways of evaluating definite integrals:

1. Use of area formulas if they are available.

   (This is what we did last lecture.)

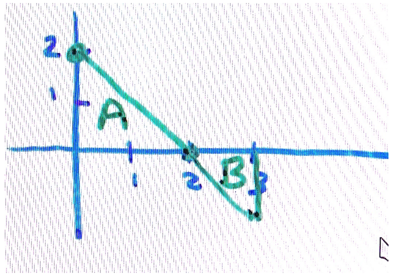
2. Use of the Fundamental Theorem of Calculus (F.T.C.)

3. Use of the Riemann sum $\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$

   (This we will not do in this course.)

Example: Evaluate $\int_0^3 (2-x)\,dx$ using the first two methods.

Solution:

1) Areas: 

$$\int_0^3 (2-x)\,dx = A - B = \frac{1}{2}\cdot 2\cdot 2 - \frac{1}{2}\cdot 1\cdot 1 = 2 - \frac{1}{2} = \frac{3}{2}$$

2) F.T.C. (No graph required)

$$\int_0^3 (2-x)\,dx = \underbrace{2x - \frac{x^2}{2}}_{\text{antideriv.}}\Big|_0^3$$

$$= (2\cdot 3 - \frac{3^2}{2}) - (2\cdot 0 - \frac{0^2}{2})$$

$$= 6 - \frac{9}{2} = \frac{3}{2}$$

Example: Evaluate using the F.T.C.

$$\int_1^8 \left(\frac{1}{\sqrt[3]{x}} - \frac{5}{x} \right) \, dx.$$

Solution:

$$\int_1^8 \left(\frac{1}{\sqrt[3]{x}} - \frac{5}{x} \right) \, dx = \int_1^8 \left(x^{-1/3} - 5x^{-1} \right) \, dx$$

$$= \left(\frac{x^{2/3}}{2/3} - 5 \ln |x| \right) \Bigg|_1^8$$

$$= \left(\frac{3}{2} \cdot 8^{2/3} - 5 \ln 8 \right) - \left(\frac{3}{2} - 5 \ln 1 \right)$$

$$= \frac{3}{2} \cdot 4 - 5 \ln 8 - \frac{3}{2}$$

$$= \frac{9}{2} - 5 \ln 8$$

Example: Evaluate using the F.T.C.

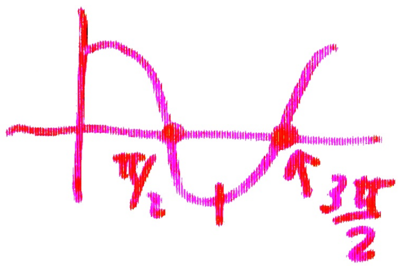
$$\int_0^{\pi/2} \sin(3x)\, dx.$$

Solution:

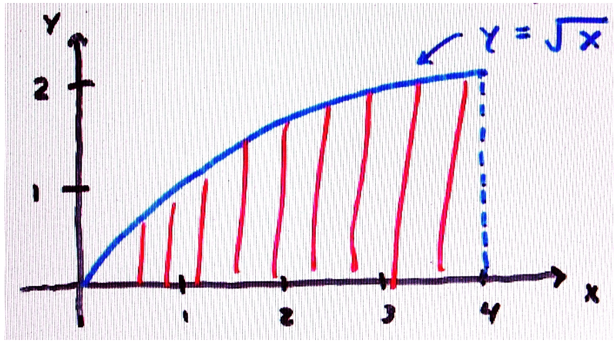
$$\int_0^{\pi/2} \sin(3x)\, dx = -\cos(3x) \cdot \frac{1}{3}\Big|_0^{\pi/2}$$

$$= -\frac{1}{3}\cos\left(\frac{3\pi}{2}\right) - \left(-\frac{1}{3}\cos(0)\right)$$

$$= 0 + \frac{1}{3} = \frac{1}{3}$$



Example: Find the area of the region below.



Solution:

$$A = \int_0^4 \sqrt{x}\, dx = \int_0^4 x^{1/2}\, dx$$

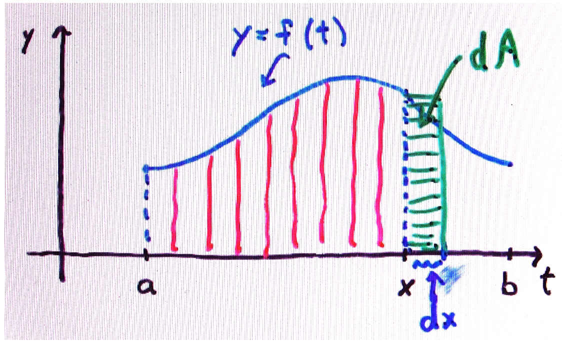
$$= \left.\frac{x^{3/2}}{3/2}\right|_0^4$$

$$= \frac{2}{3} \cdot 4^{3/2} - 0$$

$$= \frac{2}{3} \cdot 8 = \frac{16}{3}$$

Section 5.4 - Fundamental Theorem of Calculus II

Let $f(t)$ be a continuous function on $[a, b]$.



Let x be a point with $a < x < b$.

Let $A(x) = \int_a^x f(t)\,dt$ = Signed Area bounded by $f(t)$ over $[a, x]$.

Goal: Find the rate that the area $A(x)$ increases or decreases, that is, find $\frac{dA}{dx}$.

Let dx = infinitesimal change in x.

dA = resulting change in the area

dA = height $\times$ base $= f(x) \cdot dx$.

Thus: $$\frac{dA}{dx} = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

Theorem (Fundamental Theorem of Calculus II)

Let $f(x)$ be a continuous function on $[a, b]$. Then for any x in (a, b) we have

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

where $f(x)$ is the evaluation of $f(t)$ at x.

"The derivative of the integral of a function is the function."

Example: Find $\displaystyle\frac{d}{dx}\int_2^x e^t \cdot \cos(5t)\,dt$

Solution: $\displaystyle\frac{d}{dx}\int_2^x e^t \cdot \cos(5t)\,dt = e^x \cos(5x)$, by F.T.C. II.

Example: Find $\displaystyle\frac{d}{du}\int_{-3}^u \frac{1}{t^2+1}\,dt$

Solution: $\displaystyle\frac{d}{du}\int_{-3}^u \frac{1}{t^2+1}\,dt = \frac{1}{u^2+3}$, by F.T.C. II.

Flipping the limits of integration

Definition

Let $f(x)$ be a continuous function on $[a, b]$. Then

$$\int_b^a f(x)\,dx := -\int_a^b f(x)\,dx.$$

It follows that $\dfrac{d}{dx}\displaystyle\int_x^a f(t)\,dt = -f(x)$.

When variable is lower limit insert (-) sign.

Example: Find $\dfrac{d}{dx}\displaystyle\int_x^5 \sin t^2\,dt$

Solution:

$$\begin{aligned}
&\frac{d}{dx}\int_x^5 \sin t^2\,dt \\
&\quad = \frac{d}{dx}\left(-\int_5^x \sin t^2\,dt\right) \\
&\quad = -\sin x^2
\end{aligned}$$

Concept of the "dummy" variable

a) Let $F(x) = \int_a^x t^2\,dt = \left.\frac{t^3}{3}\right|_a^x = \frac{x^3}{3} - \frac{a^3}{3}$

t is a dummy variable

b) Let $F(x) = \int_a^x u^2\,du = \left.\frac{u^3}{3}\right|_a^x = \frac{x^3}{3} - \frac{a^3}{3}$

u is a dummy variable.

We see that $F(x) = G(x)$. The name of the dummy variable plays no role for the value of the integral.

Example: Find:

a) $$\frac{d}{dt}\int_3^t f(x)\,dx = f(t).$$

b) $$\frac{d}{dx}\int_3^x f(t)\,dt = f(x).$$

An extension of the F.T.C. II

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x)$$

Indeed, let $F(x) = \int_a^x f(t)\,dt$. Then the chain rule gives:

$$\frac{d}{dx}F(g(x)) = F'(g(x))\cdot g'(x) = f(g(x))\cdot g'(x).$$

Example: Find $\displaystyle\frac{d}{dx}\int_2^{x^2} \frac{dt}{\sqrt{t}+1}$.

Solution: $\displaystyle\frac{d}{dx}\int_2^{x^2} \frac{dt}{\sqrt{t}+1} = \frac{1}{\sqrt{x^2}+1}\cdot\frac{d}{dx}x^2 = \frac{2x}{x+1}$

Example: Find $\displaystyle\frac{d}{dx}\int_x^{x^3} \cos t^2\,dt$.

Solution: $\displaystyle\frac{d}{dx}\int_x^{x^3} \cos t^2\,dt = \frac{d}{dx}\int_x^{a} \cos t^2\,dt + \frac{d}{dx}\int_a^{x^3} \cos t^2\,dt$

$$= -\cos x^2 + \cos\left((x^3)^2\right)\cdot\frac{d}{dx}x^3 = 3x^2\cos x^6 - \cos x^2$$

We have seen **two ways** to find an antiderivative of $f(x)$:

1. Use our known formulas for derivatives and work backwards:

   Let $F(x)$ be such that $F'(x) = f(x)$.

2. Use a definite integral: Let

   $$A(x) = \int_a^x f(t)\,dt.$$

   Then $A'(x) = f(x)$ and $A(a) = 0$.

We have also seen that any two antiderivatives must differ by a constant. Thus:

$$A(x) = F(x) + C.$$

Let us find C:

$$A(a) = 0 = F(a) + C \quad \Rightarrow \quad C = -F(a).$$

Thus: $\quad A(x) = F(x) - F(a).$

Therefore: $$\int_a^b f(t)\,dt = A(b) = F(b) - F(a).$$

This is the F.T.C. I