Calculus I - Lecture 22 - The Definite Integral

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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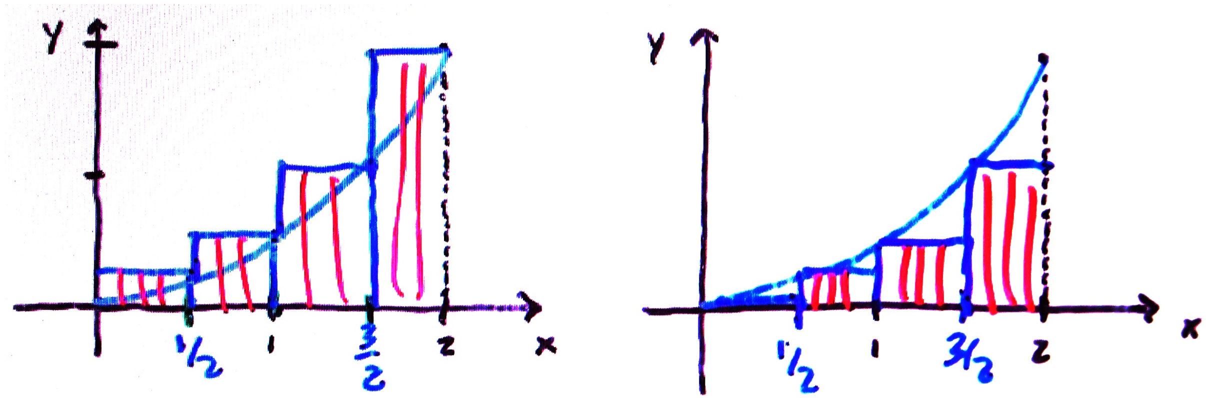
Section 5.1. — Areas below curves

Example: Estimate the area under the curve $y = x^2$ over the interval $[0, 2]$.

Solution: Divide the interval $[0, 2]$ into n equal pieces of length $2/n$ and estimate the area using rectangles.

R_n = estimate using right end-points
L_n = estimate using left end-points

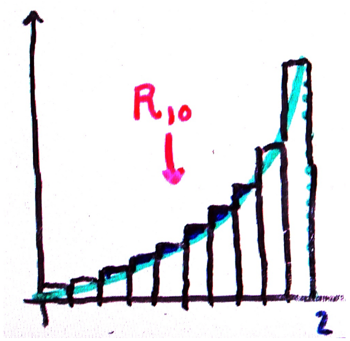
Let $n = 4$. So we have 4 equal pieces.



$$R_4 = \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{2}\cdot 1^2 + \frac{1}{2}\left(\frac{3}{2}\right)^2 + \frac{1}{2}\cdot 2^2 = 3.75$$

$$L_4 = \frac{1}{2}\cdot 0^2 + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{2}\cdot 1^2 + \frac{1}{2}\left(\frac{3}{2}\right)^2 = 1.75$$

To get more accuracy we partition the interval into more pieces.



n	R_n	L_n
4	3.75	1.75
10	3.08	2.28
100	2.7068	2.627
1000	2.6707	2.6627
10000	2.6671	2.6663
1000000	2.66667	2.66667

Actual Area $= \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = 2\frac{2}{3}$.

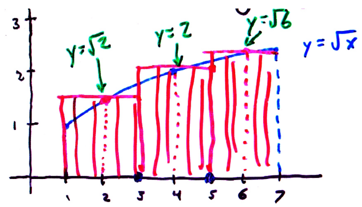
We'll see a simple way of calculating this next time using antiderivatives!

Midpoint Approximation:

M_n = Approximation of area using the midpoint of each interval.

Example: Estimate the area bounded by $y = \sqrt{x}$ over the interval $[1, 7]$ using M_3.

Solution:

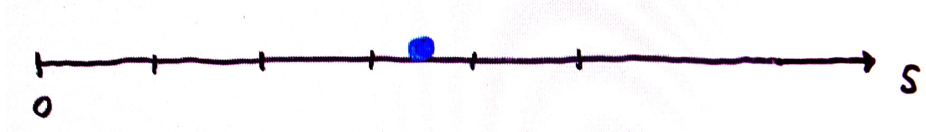


Divide the interval into 3 equal pieces:
$[1, 3]$, $[3, 5]$, $[5, 7]$.

$$M_3 = \text{sum of areas of rectangles}$$
$$= \underbrace{2 \cdot \sqrt{2}}_{\text{base}\times\text{height}} + 2 \cdot 2 + 2 \cdot \sqrt{6}$$
$$= 2 \cdot (\sqrt{2} + 2 + \sqrt{6})$$

Example: An object moves along a straight line with

$$s = s(t) = \text{position}$$
$$v = v(t) = \text{velocity}$$



Estimate the distance the object travels during the time interval $[0,6]$ if the velocity is given by

$t(\text{sec})$	0	1	2	3	4	5	6
$v(\text{m/sec})$	0	3	5	4	5	2	1

a) using R_3:Divide $[0,6]$ into 3 equal pieces and use right endpoints for distance estimate:

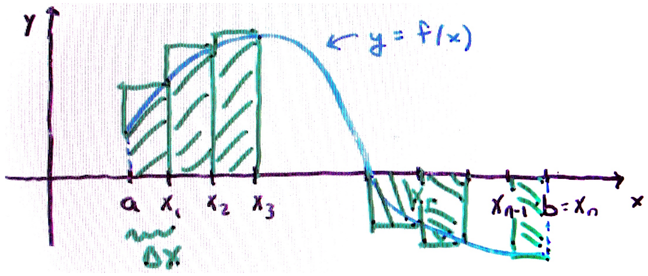
$$R_3 = \underbrace{2 \cdot 5}_{[0,2]} + \underbrace{2 \cdot 5}_{[2,4]} + \underbrace{2 \cdot 1}_{[4,6]}$$
$$= 10 + 10 + 2 = 22\,m$$

b) using M_3:Use midpoints

$$M_3 = 3 \cdot 2 + 4 \cdot 2 + 2 \cdot 2$$
$$= 6 + 8 + 4 = 18\,m$$

Section 5.2 - Riemann Sums and Definite Integrals

Let $f(x)$ be a continuous function on $[a, b]$.



Divide $[a, b]$ into n equal pieces, each of length $\Delta x = \dfrac{b-a}{n}$.

Let $x_i = a + i \cdot \Delta x$ = right end-point of the i^{th} piece.

$$R_n = \text{Riemann Sum using right end-points}$$
$$= f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + \cdots + f(x_r)\cdot\Delta x + \cdots + f(x_n)\cdot\Delta x$$
$$= \underbrace{\sum_{i=1}^{n}}_{\text{sum as } i \text{ goes from 1 to } n} \underbrace{f(x_i)\cdot\Delta x}_{i\text{-th term in sum}}$$

$\approx$ Area above x-axis bounded by $f(x)$ (where $f(x)$ is positive)
$-$ Area below x-axis bounded by $f(x)$ (where $f(x)$ is negative)

Definition

The **definite integral** *of* $f(x)$ *over the interval* $[a, b]$ *is given by*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i)\cdot \Delta x\right).$$

$$\int_a^b f(x)\,dx = \text{“Integral of } f(x) \text{ from } a \text{ to } b\text{”}$$

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i)\cdot \Delta x\right) = \text{“Limit of the Riemann sums as } n \to \infty.\text{”}$$

Theorem: The definite integral exists, that is, the above limit exists, for any continuous function on $[a, b]$

Theorem: The definite integral equals

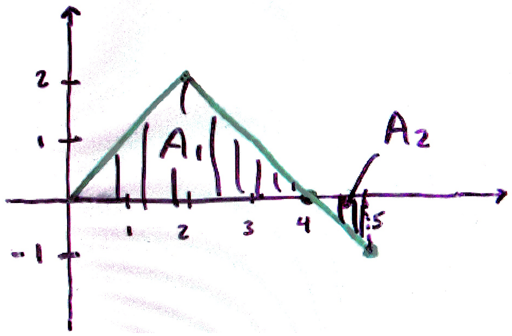
  the area above x-axis bounded by $f(x)$ over $[a, b]$
- the area below x-axis bounded by $f(x)$ over $[a, b]$

also called the “signed” area.

Example: Evaluate $\int_0^5 f(x)\,dx$ for the function below, using the signed area interpretation.

$$f(x) = \begin{cases} x, & 0 \le x \le 2 \\ 4 - x, & x > 2. \end{cases}$$

Solution:



A_1 = physical area $\qquad$ A_2 = physical area

$$\int_0^5 f(x)\,dx = A_1 - A_2$$

$$= \tfrac{1}{2} \cdot \text{base} \cdot \text{height} - \tfrac{1}{2} \cdot \text{base} \cdot \text{height}$$

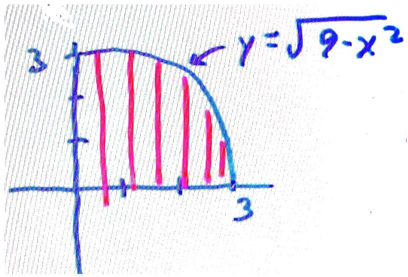
$$= \tfrac{1}{2} \cdot +4 \cdot 2 - \tfrac{1}{2} \cdot 1 \cdot 1$$

$$= 4 - \tfrac{1}{2} = \tfrac{7}{2}$$

Example: Evaluate using signed areas:

$$\int_0^3 \sqrt{9 - x^2}\, dx.$$

Solution:



$$y = \sqrt{9 - x^2} \Rightarrow y^2 = 9 - x^2 \Rightarrow x^2 + y^2 = 9 = 3^2$$

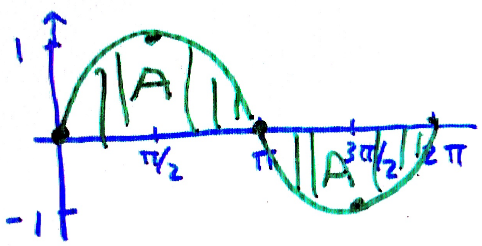
$$\int_0^3 \sqrt{9 - x^2}\, dx = \text{Area of quarter circle of radius 3}$$

$$= \frac{1}{4}(\pi \cdot 3^2) = \frac{9}{4}\pi$$

Example: Evaluate using signed areas:

$$\int_0^{2\pi} \sin x \, dx.$$

Solution:



$$\int_0^{2\pi} \sin x \, dx = \text{Signed Area}$$
$$= A - A$$
$$= 0$$

Properties of the definite integral

1) **Constant factor rule:**

$$\int_a^b c \cdot f(x)\,dx = c \cdot \int_a^b f(x)\,dx$$

2) **Sum & difference rule:**

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

3) **Additivity rule:**

Suppose $a < b < c$. Then

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

Example: Given that $\int_0^a x^2\,dx = \frac{a^3}{3}$, for any $a > 0$, find:

a) $\int_0^3 7x^2\,dx$,

b) $\int_1^5 x^2\,dx$.

Solution:

a) $\int_0^3 7x^2\,dx = 7 \cdot \int_0^3 x^2\,dx$ (using rule 1)

$= 7 \cdot \frac{3^3}{3} = 63$ (using given formula)

b) $\int_1^5 7x^2\,dx = \int_0^5 x^2\,dx - \int_0^1 x^2\,dx$ (using rule 3)

$= \frac{5^3}{3} - \frac{1^3}{3} = \frac{124}{3}$ (using given formula)