Calculus I - Lecture 21 - Review Exam 3

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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Example: Use logarithmic differentiation to find $\frac{dy}{dx}$ for

$$y = x^x(x^2+1)^{5/2}.$$

Solution:

$$\begin{aligned}
\ln y &= \ln(x^x) + \ln\left((x^2+1)^{5/2}\right)\\
\ln y &= x\ln x + \frac{5}{2}\ln(x^2+1)\\
\frac{\mathrm{d}}{\mathrm{d}x}\ln y &= \frac{\mathrm{d}}{\mathrm{d}x}(x\ln x) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{5}{2}\ln(x^2+1)\right)\\
\frac{1}{y}\cdot y' &= (\ln x + x\cdot\frac{1}{x}) + \frac{5}{2}\frac{1}{x^2+1}2x\\
y' &= x^x(x^2+1)^{5/2}\left[\ln x + 1 + \frac{5x}{x^2+1}\right]
\end{aligned}$$

Example: Find the tangent line to the curve $x^2y^2 = y^2 + 3$ at $(2, 1)$.

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2y^3) = \frac{\mathrm{d}}{\mathrm{d}x}(y^2 + 3)$$

$$2x \cdot y^3 + x^2 \cdot 3y^2 \cdot y' = 2y \cdot y'$$

$$x^2 \cdot 3y^2 \cdot y' - 2y \cdot y' = -2x \cdot y^3$$

$$(3x^2 \cdot y^2 - 2y)y' = -2x \cdot y^3$$

$$y' = \frac{-2x \cdot y^3}{3x^2y^2 - 2y}$$

$$y'\Big|_{(2,1)} = \frac{-2 \cdot 2 \cdot 1^3}{3 \cdot 2^2 \cdot 1^2 - 2 \cdot 1} = \frac{-4}{10} = -\frac{2}{5}$$

Tangent line: $y - 1 = -\frac{2}{5}(x - 2)$ or $y = -\frac{2}{5}x + \frac{9}{5}$

Example: Find the tangent line of $f(x) = \sqrt[3]{x}$ at $x = 27$ and use use it to approximate $\sqrt[3]{27.1}$

Solution:

$f(x) = x^{1/3}$, $\qquad f(27) = 3$

$f'(x) = \frac{1}{3}x^{-2/3}$, $\qquad f'(27) = \frac{1}{3} \cdot (27)^{-2/3} = \frac{1}{3 \cdot 3^2} = \frac{1}{27}$.

Tangent line: $L(x) = f(a) + f'(a)(x - a)$

$$L(x) = 3 + \frac{1}{27}(x - 27)$$

$$\sqrt[3]{27.1} = f(27.1) \approx L(27.1) = 3 + \frac{1}{27}(27.1. - 27) = 3 + \frac{1}{270}.$$

Example: Grain flows into a conical pile such that the height increases 2 ft/min while the radius increases 3 ft/min. At what rate is the volume increasing when the pile is 2 feet high and has a radius of 4 feet.

Solution:

Volume of the cone: $V = \frac{1}{3}\pi r^2 h$

$\frac{\mathrm{d}h}{\mathrm{d}t} = 2$ ft/min when $h = 2$

$\frac{\mathrm{d}r}{\mathrm{d}t} = 3$ ft/min when $r = 4$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{3}\pi r^2 h\right) = \frac{1}{3}\pi\left[2r \cdot \frac{\mathrm{d}r}{\mathrm{d}t} \cdot h + r^2 \cdot \frac{\mathrm{d}h}{\mathrm{d}t}\right]$$

Evaluating at $h = 2$ and $r = 4$ gives:

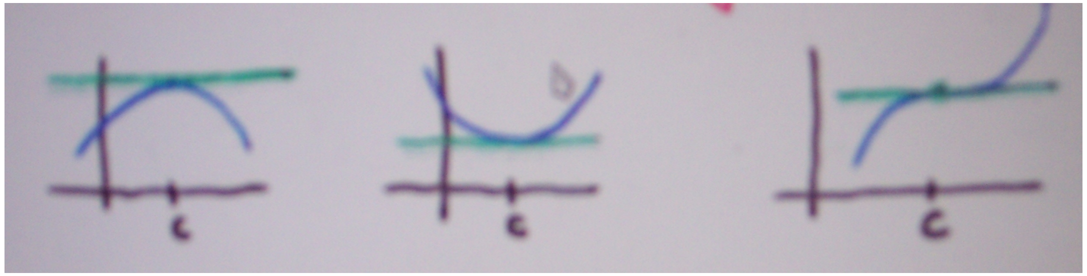
$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{3}\pi\left[2 \cdot 4 \cdot 3 \cdot 2 + 4^2 \cdot 2\right] = \frac{1}{3}\pi\,[48 + 82]$$

$$= \frac{80\pi}{3}\ \mathrm{ft}^3/\mathrm{sec}.$$

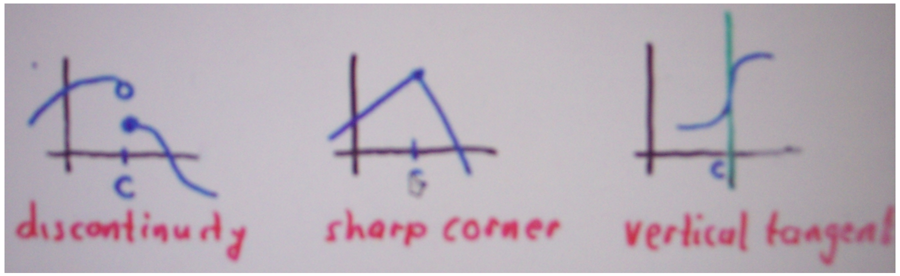
Definition

A point c in the domain of a function $f(x)$ is called a critical point if either

1. $f'(c) = 0$ *(horizontal tangent)*



2. *or $f'(c)$ does* **not** *exist.*



Theorem
If $f(c)$ is a local maximum or minimum, then c is a critical point of $f(x)$.

Theorem
Suppose that $f(x)$ is continuous on the closed interval $[a, b]$. Then $f(x)$ attains its absolute maximum and minimum values on $[a, b]$ at either:

- *A critical point*
- *or one of the end points a or b.*

Example:

a) Find the critical points for the function $f(x) = 3x - x^3$.

b) Find the maximal and minimal values of $f(x) = 3x - x^3$ on the interval $[-1, 3]$.

Solution:

a) $f'(x) = 3 - 3x^2 = 3(1 - x^2)$

$f'(x) = 0$ when $x^2 = 1$ or $x = \pm 1$.

$f'(x)$ exists for all real numbers.

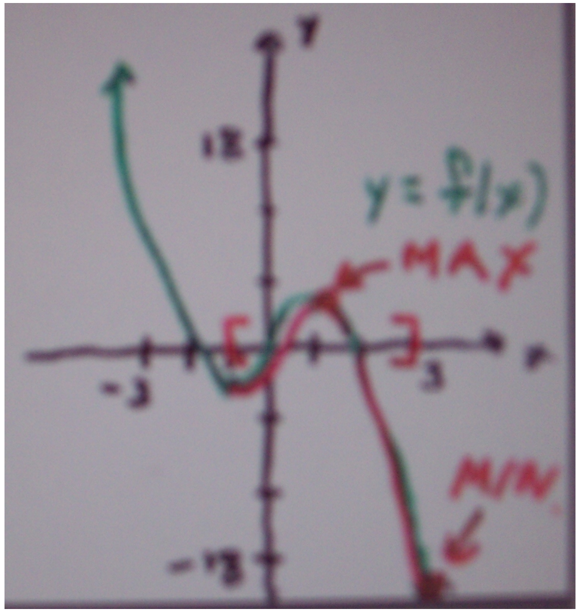
The only critical points are ± 1.

b) We make a table with the critical points inside the interval and its endpoints.

x	$3x - x^3$
1	$3 - 1 = 2$
-1	$-3 - (-1) = -2$
3	$3 \cdot 3 - 3^3 = -18$

Maximal value is 2 at $x = 1$,

Minimal value is -18 at $x = 3$.


y
y = f(x)
MAX
MIN
-3
3

Recall:

Critical points: Points c in the domain of $f(x)$ where $f'(c)$ does not exist or $f'(c) = 0$.

Monotony:

$f'(x) > 0 \Rightarrow f(x)$ increasing: If $a < b$ then $f(a) < f(b)$
$f'(x) < 0 \Rightarrow f(x)$ decreasing: If $a < b$ then $f(a) > f(b)$

Local extrema: Appear at points c in the domain of $f(x)$ where $f(x)$ changes from increasing to decreasing ($f(c)$ maximum) or from decreasing to increasing ($f(c)$ minimum).

First derivative test:

Let $f'(c) = 0$. Then:

$f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c \Rightarrow f(c)$ is local max.
$f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c \Rightarrow f(c)$ is local min.

Concavity:

$f''(x) > 0 \Rightarrow f(x)$ concave up: $f'(x)$ increasing:
$f''(x) < 0 \Rightarrow f(x)$ concave down: $f'(x)$ decreasing:

Inflection points: Points $(x, f(x))$ where the graph of $f(x)$ changes its concavity.

Inflection point test:

Let $f''(c) = 0$.
If $f''(x)$ changes its sign at $x = c$ then $f(x)$ has a inflection point at $x = c$.

Second derivative test:

Let $f'(c) = 0$. Then:

$f''(c) < 0 \Rightarrow f(c)$ is a local maximum
$f''(c) > 0 \Rightarrow f(c)$ is a local minimum

Transition points: Points where $f'(x)$ or $f''(x)$ has a sign change.

Graph Sketching

Main Steps

1. Determine then domain.
2. Find points with $f'(x) = 0$ and mark sign of $f'(x)$ on number line.
3. Find points with $f''(x) = 0$ and mark sign of $f''(x)$ on number line.
4. Compute function values for transition points.
5. Find asymptotes.
6. Sketch graph.
7. Take a break.

Example: Sketch the graph of $y = x(8-x)^{1/3}$.

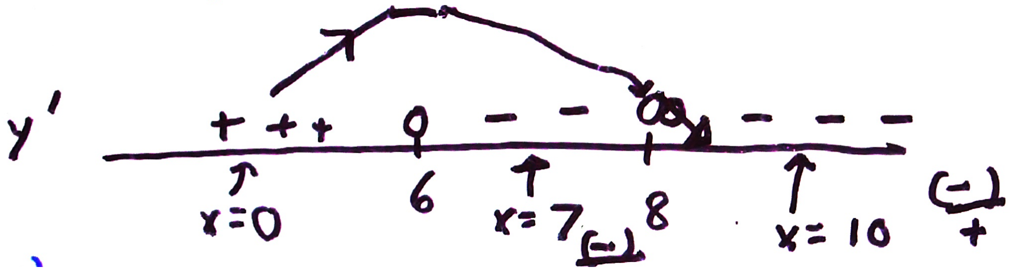
Solution:

1. All real numbers.

2. $y' = \ldots = \frac{24-4x}{3(8-x)^{2/3}}$

$y' = 0 \Leftrightarrow 24 - 4x = 0 \Leftrightarrow x = 6$

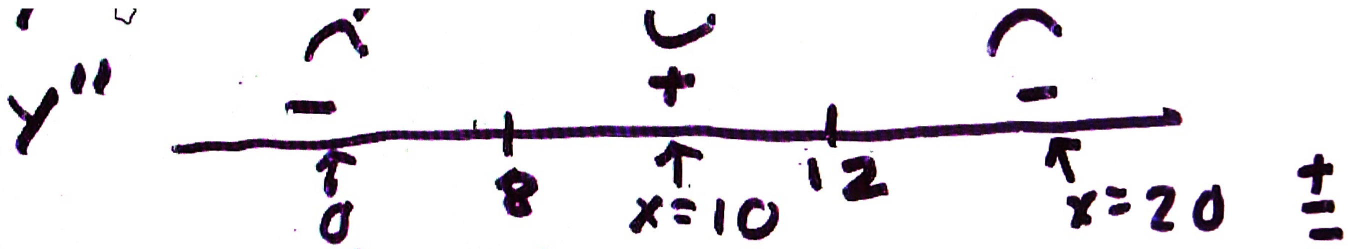
y'D.N.E $\Leftrightarrow 8 - x = 0 \Leftrightarrow x = 8$



3. $y'' = \ldots = \frac{4x-48}{9(8-x)^{5/3}}$

$y'' = 0 \Leftrightarrow 4x - 48 = 0 \Leftrightarrow x = 12$

y'' D.N.E. $\Leftrightarrow 8 - x = 0 \Leftrightarrow x = 8$

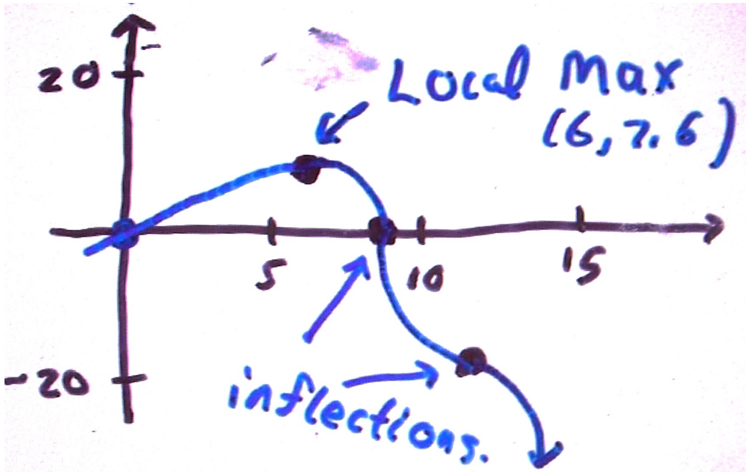


4.

x	y
6	7.6
8	0
12	-19.0
0	0

5. No horizontal (limit for $x \longrightarrow \pm\infty$ D.N.E.) or vertical asymptotes ($f(x)$ is everywhere defined).

6.



7. No time!

7-**Step Procedure for Applied Maximum/Minimum Problems:**

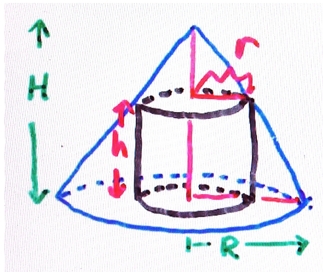
1. Draw Picture, label variables.
2. Restate the problem:
   - What is given?
   - Which variable should be maximized or minimized?
3. Find the relationship between variables:

   Geometric Formula, Trigonometric equation, Pythagorean theorem, etc.
4. Express the quantity being maximized or minimized in terms of a single variable.
5. Find the critical points ($f'(x) = 0$ or not defined).
6. Find the absolute Minima or Maxima.
7. Compute the remaining variables (if asked for) and state the answer in a sentence.

Example: A right circular cylinder is inscribed in a right circular cone. Find the dimensions that maximize the volume of the cylinder.

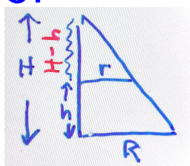
Solution:

1.



2. $H =$ height of cone, $R =$Radius of cone (given constants)
$h =$ height of cylinder, $r =$radius of cylinder (asked for)
Maximize: Volume cylinder $= V = \pi r^2 h$

3.



Similar triangles: $\dfrac{r}{R} = \dfrac{H-h}{H} \Rightarrow r = \dfrac{R}{H}(H-h)$

4.

$$V = \pi \frac{R^2}{H^2}(H-h)^2 h$$

5. $\dfrac{dV}{dh} = \pi \dfrac{R^2}{H^2} \left[2(H-h)(-1)h + (H-h)^2\right] = 0$

$\Leftrightarrow (H-h)\left[-2h + (H-h)\right] = (H-h)\left[H-3h\right] = 0$

$\Leftrightarrow h = H$ or $H - 3h = 0$, $h = \frac{1}{3}H$.

6. Since $V = 0$ for $h = 0$ or $h = H$ and $V > 0$ for h between, $h = \frac{1}{3}H$ is the absolute maximum.

7. $r = \frac{R}{H}(H - \frac{1}{3}H) = \frac{2}{3}R$.

The volume of the cylinder is maximized for the height $h = \frac{1}{3}H$ and radius $r = \frac{2}{3}R$.