Calculus I - Lecture 20 - The Indefinite Integral

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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Reminder: Exam 3 on Thursday, April 10.

Review in Wednesday's Lecture

Practice Exam online on course homepage

Recall that there are two main parts of Calculus

1. **Derivatives:** Measures instantaneous change
2. **Integrals:** Measures cumulative amounts

We are now ready to begin part 2. It begins with the study of the reverse operation of taking the derivative.

Definition (Antiderivative)

A primitive or antiderivative of a function $f(x)$ is function $F(x)$ such that $F'(x) = f(x)$.

Example: Find an antiderivative of x^3, by trial and error.

Solution: Initial guess: x^4 (since derivation decreases the degree of a power function by 1):

$$\frac{\mathrm{d}}{\mathrm{d}x} x^4 = 4\,x^3.$$

Thus: $\frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{4}x^4) = \frac{1}{4}(4x^3) = x^3$.

Note: $\frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{4}x^4 - 7) = x^3$

All functions $F(x) = \frac{1}{4}x^4 + C$, C any constant, are antiderivatives.

Did we find all antiderivatives?

Theorem

Let $F(x)$ be an antiderivative of the function $f(x)$ defined on (a, b). Then any antiderivative on (a, b) of $f(x)$ is of the form $F(x) + C$ for some constant C.

Proof: Let $G(x)$ be another antiderivative of $F(x)$. Set $H(x) = G(x) - F(x)$. Then

$$H'(x) = G'(x) - F'(x) = f(x) - f(x) = 0.$$

We claim that $H(x)$ must be a constant function. For, if it would be not, there exist (at least) two points $x = u$ and $x = v$ in (a, b) with $H(u) \neq H(v)$. By the mean value theorem there exists then a point $x = c$ in (u, v) such that

$$\frac{H(u) - H(v)}{u - v} = H'(c).$$

But since $H(u) \neq H(v)$ this would mean $H'(c) \neq 0$, a contradiction. Thus $H(x) = C$ for some constant C. This implies $G(x) = F(x) + C$. q.e.d.

Definition (Indefinite Integral)

The indefinite integral or general antiderivative $\int f(x)\mathrm{d}x$ *of a function* $f(x)$ *stands for all possible antiderivatives of* $f(x)$ *defined on an interval, i.e.*

$$\int f(x)\,\mathrm{d}x = F(x) + C, \text{ where } C \text{ is a constant}$$

and $F(x)$ *is an arbitrary antiderivative of* $f(x)$.

Notation: In the expression $\displaystyle\int f(x)\mathrm{d}x$, the function $f(x)$ is called the **integrand** and $\mathrm{d}x$ is a differential (in its symbolic meaning). The constant C as above is called the **constant of integration**.

The indefinite integral should not be confused with the **definite integral** $\int_a^b f(x)\,\mathrm{d}x$ which we will consider next week and is defined as a limit of a sum. The symbol $\int$ is a stretched **S** and reminds about the **S**um. We will also explain the relation between the indefinite and the definite integral.

Power Rule: The indefinite integral of a power function $f(x) = x^n$, where $n \neq -1$ is

$$\int x^n \, \mathrm{d}x = \frac{1}{n+1} x^{n+1} + C.$$

Raise the exponent by 1 and divide by the raised exponent.

Example: Find the indefinite integral of the following functions:

a) $f(x) = x^{13}$ $\quad \int f(x) \, \mathrm{d}x = \frac{x^{14}}{14} + C$

b) $f(x) = \sqrt{x} = x^{1/2}$ $\quad \int f(x) \, \mathrm{d}x = \frac{x^{3/2}}{3/2} + C = \frac{2x^{3/2}}{3} + C$

c) $f(x) = \frac{1}{x^3} = x^{-3}$ $\quad \int f(x) \, \mathrm{d}x = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$

d) $f(x) = 1 = x^0$ $\quad \int f(x) \, \mathrm{d}x = x + C$

Table of Indefinite Integrals

$f(x)$	$\int f(x)\mathrm{d}x$
1	$x + C$
x^n	$\dfrac{x^{n+1}}{n+1} + C, \;\; n \neq 1$
$\dfrac{1}{x}$	$\ln\lvert x\rvert + C$
e^x	$e^x + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$

$f(x)$	$\int f(x)\mathrm{d}x$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$
a^x	$\dfrac{1}{\ln a} a^x + C$
$\dfrac{1}{1+x^2}$	$\arctan x + C$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$

Proof by derivation.

Guess and Fudge Method

Example: Find an antiderivative of $\cos(3x)$.

Solution:
Since $\int \cos x \mathrm{d}x = \sin x + C$ we try $\sin(3x)$ with fudge factor $\frac{1}{3}$:
$\frac{1}{3}\sin(3x)$. Indeed $(\frac{1}{3}\sin(3x))' = \frac{1}{3}\cos(3x)\cdot 3 = \cos(3x)$.
So $\frac{1}{3}\sin(3x)$ is an antiderivative.

The guess and fudge method applies to functions of the form $f(ax+b)$, where a and b are constants.

$$\int f(ax+b)\mathrm{d}x = \frac{1}{a}F(ax+b) + C$$

where $F(x)$ is an antiderivative of $f(x)$.

Example:

a) $\displaystyle\int \sin(2x-\pi)\mathrm{d}x = -\frac{1}{2}\cos(2x-\pi) + C$

b) $\displaystyle\int e^{5-3x}\mathrm{d}x = -\frac{1}{3}e^{5-3x} + C$

Rules for the indefinite integral

1) Constant factor rule:

$$\int k \cdot f(x)\,\mathrm{d}x = k \cdot \int f(x)\,\mathrm{d}x$$

Proof: $(kF(x))' = k \cdot F'(x)$.

2) Sum and difference rule:

$$\int (f(x) \pm g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x \pm \int g(x)\,\mathrm{d}x$$

Proof: $(F(x) \pm G(X))' = F'(x) \pm G'(x)$.

Example: Find $\int (e^{3x} + 7x^{-1})\,\mathrm{d}x$.

Solution:

$$= \int e^{3x}\,\mathrm{d}x + 7\int x^{-1}\,\mathrm{d}x \text{ by rule 1) and 2)}$$

$$= \frac{1}{3}e^{3x} + 7\ln|x| + C$$

Example: Find $\int \left(\frac{1}{x-2} + (3x+7)^5\right)\,\mathrm{d}x$.

Solution:

$$= \ln|x-2| + \frac{(3x+7)^6}{6\cdot 3} + C$$

Example: Find $\int \frac{\mathrm{d}x}{1+x^2}$

Solution:

$$= \int \left(\frac{1}{1+x^2}\right)\,\mathrm{d}x$$

$$= \arctan x + C$$

Application to differential equations

Example: Find a function $f(x)$ such that

$$f'(x) = 6x(1-x) \quad \text{and} \quad f(0) = 1.$$

Solution:

$f(x)$ is an antiderivative of $6x(1-x)$. Thus:

$$f(x) = \int 6x(1-x)\,\mathrm{d}x$$

$$= \int (6x - 6x^2)\,\mathrm{d}x$$

$$= 6 \cdot \frac{x^2}{2} - 6 \cdot \frac{x^3}{3} + C$$

$$= 3x^2 - 2x^3 + C$$

When $x = 0$: $f(0) = 1$

$$\Leftrightarrow 3 \cdot 0 - 2 \cdot 0 + C = 1 \Leftrightarrow C = 1.$$

$$f(x) = 3x^2 - 2x^3 + 1$$

Example: A body falls to the ground. During the fall, it feels a constant acceleration of g where $g = 32\,\mathrm{ft}/\mathrm{sec}^2$. At time $t = 0$ the body has the height y_0 and the velocity v_0. Find a formula for the the height y in terms of t.

Solution:

Let $y = y(t)$ be the height function, $v = v(t) = \frac{dy}{dt}$ be the velocity function and $a = a(t) = \frac{dv}{dt}$ be the acceleration function.

We have $a(t) = -g$ (downward acceleration).

Since v is an antiderivative of $a(t)$ one has:

$$v = \int -g\,\mathrm{d}t = -g\int 1\,\mathrm{d}t = -gt + C$$

$$v(0) = v_0 \;\Rightarrow\; 0 + C = v_0 \;\Rightarrow\; C = v_0$$

Thus: $v = -gt + v_0$.

Since y is an antiderivative of $v(t)$ one has:

$$y = \int (-gt + v_0)\,\mathrm{d}t = -g\frac{t^2}{2} + v_0 t + C$$

$$y(0) = y_0 \;\Rightarrow\; 0 + 0 + C = y_0 \;\Rightarrow\; C = y_0$$

Thus: $y = g\frac{t^2}{2} + v_0 t + y_0$.