Calculus I - Lecture 19 - Applied Optimization

Lecture Notes:
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:
http://www.math.ksu.edu/math220/spring-2014/indexs14.html

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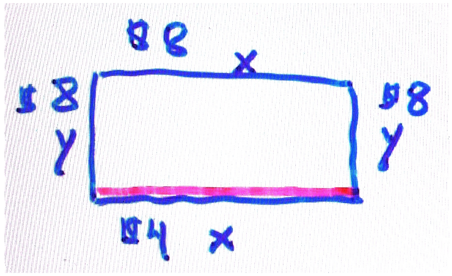
7-**Step Procedure for Applied Maximum/Minimum Problems:**

1. Draw Picture, label variables.
2. Restate the problem:
   - What is given?
   - Which variable should be maximized or minimized?
3. Find the relationship between variables:

   Geometric Formula, Trigonometric equation, Pythagorean theorem, etc.
4. Express the quantity being maximized or minimized in terms of a single variable.
5. Find the critical points ($f'(x) = 0$ or not defined).
6. Find the absolute Minima or Maxima.
7. Compute the remaining variables (if asked for) and state the answer in a sentence.

Example: A rectangular garden of area 75 ft^2 is bounded on three sides by a fence costing \$8 per feet and on the 4^{th} side by a fence costing \$4 per feet. Find the dimensions that will minimize the total cost.

Solution:

1. 

2. Given: Area $= A = 75\mathrm{ft}^2$, cost of two types of fences per feet. Minimize: cost of fence $= C$.

3. $A = xy = 75, \qquad C = 4x + 8y + 8y + 8x = 12x + 16y.$

4. $y = \dfrac{75}{x}$, so $C = 12x + 16 \cdot \dfrac{75}{x} = 12x + 1200 \cdot x^{-1}$

5. $$\frac{dC}{dx} = 12 - 1200x^{-2} = 0 \Leftrightarrow 12 = 1200x^{-2} \Leftrightarrow x^2 = 100 \Leftrightarrow x = 10$$

6. Since $\lim_{x \to 0} C = \infty$ and $\lim_{x \to \infty} C = \infty$, the critical point $x = 10$ must be a global minimum.

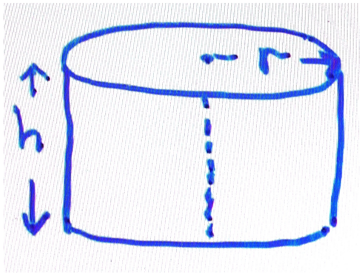
7. $y = \frac{75}{x} = 7.5$

The total cost of the fence is minimized by a garden length of 10 feet and a width of 7.5 feet.

Example: A canning company wishes to design a can of a volume of 100 cm^3 using the least amount of metal as possible. Find the dimensions it should use.

Solution:

1.



2. Given: Volume $V = 100\ \mathrm{cm}^3$.
Minimize: surface area $= A$

3. $A = \underbrace{2\pi r^2}_{\text{top \& bottom}} + \underbrace{h \cdot 2\pi r}_{\text{wall}} = 2\pi r^2 + 2\pi hr.$

$V = \pi r^2 h = 100 \Rightarrow h = \dfrac{100}{\pi r^2}$

4. $A = 2\pi r^2 + 2\pi \cdot \dfrac{100}{\pi r^2} \cdot r = 2\pi r^2 + 200 \cdot r^{-1}.$

5. $\dfrac{dA}{dr} = 4\pi r - 200 \cdot r^{-2} = 0 \Leftrightarrow 4\pi r = \dfrac{200}{r^2} \Leftrightarrow \pi r^3 = 50$
$\Leftrightarrow r = \sqrt[3]{50/\pi}$

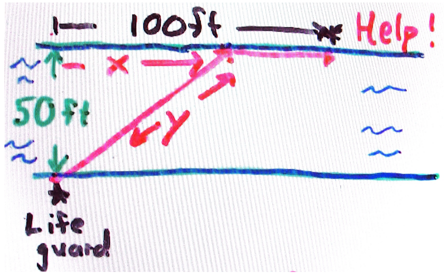
6. Since $\lim_{r \to 0} A = \infty$ and $\lim_{r \to \infty} A = \infty$, the critical point $r = \sqrt[3]{50/\pi}$ must be a global minimum.

7. $h = \dfrac{100}{\pi r^2} = \dfrac{100}{\pi} \cdot (50/\pi)^{-2/3} = 2\sqrt[3]{50/\pi}$
The total amount of metal is minimized by a height of $2\sqrt[3]{50/\pi} \approx 5.030$ cm and a radius of $\sqrt[3]{50/\pi} \approx 2.515$ cm.

Example: A lifeguard wishes to get to a person 100 ft downstream on the opposite shore of a 50 ft wide river, as fast as possible. What path should she take if she can swim 5 ft/sec and run 15 ft/sec.

Solution:

1. We can assume that she swims in a straight line to a point x ft downstream.



2. Given: width of the river $50\mathrm{ft}$, distance of person along river, velocity of lifeguard on land and in water.
Minimize: time to get to person $= T$.

3. $T = T_{\mathrm{swim}} + T_{\mathrm{run}}$, $\mathrm{velocity} = \frac{\mathrm{Distance}}{\mathrm{time}} \Rightarrow \mathrm{time} = \frac{\mathrm{Distance}}{\mathrm{velocity}}$
$T_{\mathrm{swim}} = \frac{y}{5}$, $T_{\mathrm{run}} = \frac{100-x}{15}$, $x^2 + 50^2 = y^2$.

4.
$T = \frac{1}{5}\sqrt{x^2+50^2} + \frac{1}{15}(100-x)$

5. $$\frac{dT}{dx} = \frac{1}{5} \cdot \frac{1}{2} \frac{1}{\sqrt{x^2 + 50^2}} \cdot 2x + \frac{1}{15}(-1) = 0$$

$$\Leftrightarrow \frac{x}{5\sqrt{x^2+50^2}} = \frac{1}{15} \Leftrightarrow 3x = \sqrt{x^2+50^2}$$

$$\Leftrightarrow 9x^2 = x^2 + 50^2 \Leftrightarrow 8x^2 = 50^2 \Leftrightarrow x = \frac{50}{\sqrt{8}} = \frac{25}{\sqrt{2}}$$

6.

x	T
0	16.66
$25/\sqrt{2}$	16.094
100	22.36

$x = \frac{25}{\sqrt{2}}$ is global minimum.

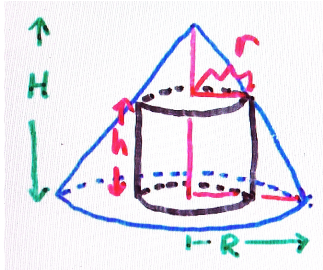
7. Life guard should swim straight to the point $x = \frac{25}{\sqrt{2}} \approx 17.68$ and then run to the person.

Mathematics Saves Life!

Example: A right circular cylinder is inscribed in a right circular cone. Find the dimensions that maximize the volume of the cylinder.

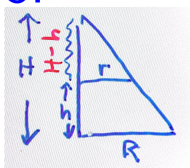
Solution:

1.



2. $H =$ height of cone, $R =$Radius of cone (given constants)
$h =$ height of cylinder, $r =$radius of cylinder (asked for)
Maximize: Volume cylinder $= V = \pi r^2 h$

3.



Similar triangles: $\dfrac{r}{R} = \dfrac{H-h}{H} \Rightarrow r = \dfrac{R}{H}(H-h)$

4.

$V = \pi \dfrac{R^2}{H^2}(H-h)^2 h$

5. $\dfrac{dV}{dh} = \pi \dfrac{R^2}{H^2} \left[2(H-h)(-1)h + (H-h)^2\right] = 0$

$\Leftrightarrow (H-h)\left[-2h + (H-h)\right] = (H-h)\left[H-3h\right] = 0$

$\Leftrightarrow h = H$ or $H - 3h = 0$, $h = \frac{1}{3}H$.

6. Since $V = 0$ for $h = 0$ or $h = H$ and $V > 0$ for h between, $h = \frac{1}{3}H$ is the absolute maximum.

7. $r = \frac{R}{H}(H - \frac{1}{3}H) = \frac{2}{3}R$.

The volume of the cylinder is maximized for the height $h = \frac{1}{3}H$ and radius $r = \frac{2}{3}R$.